COTORSION PAIRS FOR BEXT AND A GENERALIZATION OF WHITEHEAD'S PROBLEM

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Nicole Hülsmann

aus Essen

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Gutachter:	Prof. Dr. R. Göbel
	Prof. Dr. L. Salce

Nicht weil es unerreichbar ist, wagen wir es nicht,
sondern weil wir es nicht wagen, ist es unerreichbar.
(Seneca)

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Dedicated to my father.

Contents

Introduction

In 1979 Salce [Sa] introduced the notion of cotorsion pairs. A pair $(\mathcal{G},\mathcal{H})$ of classes of abelian groups is called a cotorsion pair if $\mathcal{G}$ and $\mathcal{H}$ are maximal with respect to the property that $\mathrm{Ext}(G,H)=0$ for all $G\in\mathcal{G}$ and $H\in\mathcal{H}$. A cotorsion pair $(\mathcal{G},\mathcal{H})$ is called generated by a class $\mathcal{A}$ of abelian groups if $({}^{\perp}(\mathcal{A}^{\perp}),\mathcal{A}^{\perp})=(\mathcal{G},\mathcal{H})$, where $\mathcal{A}^{\perp}=\{X\,|\,\mathrm{Ext}(A,X)=0$ for all $A\in\mathcal{A}\}$ and ${}^{\perp}\mathcal{A}=\{Y\,|\,\mathrm{Ext}(Y,A)=0$ for all $A\in\mathcal{A}\}$. Likewise, a cotorsion pair $(\mathcal{G},\mathcal{H})$ is called cogenerated by a class $\mathcal{A}$ of abelian groups if $({}^{\perp}\mathcal{A},({}^{\perp}\mathcal{A})^{\perp})=(\mathcal{G},\mathcal{H})$. A partial ordering of the cotorsion pairs is defined by $(\mathcal{G},\mathcal{H})\le(\mathcal{G}',\mathcal{H}')$ iff $\mathcal{G}\subseteq\mathcal{G}'$ for two cotorsion pairs $(\mathcal{G},\mathcal{H})$ and $(\mathcal{G}',\mathcal{H}')$. Salce defined the ordering conversely ($(\mathcal{G},\mathcal{H})\le(\mathcal{G}',\mathcal{H}')$ iff $\mathcal{G}'\subseteq\mathcal{G}$) but, of course, his results hold mutatis mutandis for this ordering. He showed that the cotorsion pairs form a complete lattice and he proved that every cotorsion pair has enough projectives if and only if it has enough injectives. Moreover, he showed that there is a bijection from the set of all cotorsion pairs between the classical cotorsion pair (the pair generated by $\mathbb{Q}$) and the maximal one (the pair generated by the class of all abelian groups) to the power set of the set of all primes. Salce started a characterization of the groups A such that $\mathrm{Ext}(R,A)=0$ for a rational group $R\subseteq\mathbb{Q}$. With the help of these results Göbel, Shelah and Wallutis [GSW] showed that the sublattice of all cotorsion pairs singly generated by a rational group is isomorphic to the lattice of all types in the sense of Baer [B], i.e. isomorphism classes of rank 1 groups. Furthermore, they proved that there is an embedding from any power set into the lattice of all cotorsion pairs. Hence there are ascending and descending chains as well as anti–chains of arbitrary length in the lattice of all cotorsion pairs. For the proof of this embedding Göbel, Shelah and Wallutis used an important result due to Eklof and Trlifaj [ET]. For every module B over any ring Eklof and Trlifaj constructed a related module A such that $\mathrm{Ext}(B,A)=0$. This construction can also be used to obtain splitters, i.e. modules A such that $\mathrm{Ext}(A,A)=0$. With the help of these results Bican, El Bashir and Enochs [BEE] proved the flat cover conjecture, namely that every module has a flat cover. This question had been open for a long time. In Chapter 3 we will transfer these results to the functor Bext. The functor Bext is defined as a subfunctor of Ext, where the group $\mathrm{Bext}(C,A)$ is the subgroup of $\mathrm{Ext}(C,A)$ that consists of all balanced–exact sequences

$$0\to A\to B\to C\to 0.$$

Balanced–exact sequences of arbitrary abelian groups were defined by Hunter [Hu] in 1976. Hunter characterized the balanced–injective groups and the balanced–projective groups. A group G is balanced–injective if and only if it is pure–injective. The torsion–free balanced–

projective groups are exactly the completely decomposable groups. Moreover, he showed that there are enough balanced–projectives but not enough balanced–injectives, i.e. every group G is an epimorphic image of a balanced–projective group with balanced kernel but not every group can be embedded as a balanced subgroup into a balanced–injective group. We will present the most important steps of Hunter's proof in Sections 2.2 and 2.3.

The functor Bext plays an important role in the theory of Butler groups. The definition of Butler groups of finite rank goes back to M. C. R. Butler [B] in 1965, who defined them as pure subgroups of completely decomposable groups of finite rank. Bican and Salce [BS] showed in 1983 that a torsion–free group G of finite rank is Butler if and only if $\mathrm{Bext}(G, T) = 0$ for all torsion groups T. This led them to one of the possible generalizations of Butler groups to groups of arbitrary rank. A torsion–free group G of arbitrary rank is called a B_1–group if $\mathrm{Bext}(G, T) = 0$ for all torsion groups T. Bican and Salce also gave another possible generalization of Butler groups, the B_2–groups. A torsion–free group G is called a *B_2–group* if there is a continuous well–ordered ascending chain of pure subgroups

$$0 = G_0 < G_1 < \cdots < G_\kappa = G = \bigcup_{\alpha<\kappa} G_\alpha$$

such that for all $\alpha < \kappa$ the quotient $G_{\alpha+1}/G_\alpha$ is of rank 1 and $G_{\alpha+1} = B_\alpha + G_\alpha$ for some finite rank Butler group B_α. They proved that every B_2–group is a B_1–group and that for countable groups C these two definitions coincide. The answer to the question if the classes of B_1–groups and B_2–groups coincide depends on the underlying set theory. In [FM] Fuchs and Magidor showed that in Gödel's universe L every B_1–group is a B_2–group and hence both classes are the same. Using Cohen forcing Shelah and Strüngmann [SS] constructed in 2003 a model of set theory in which a B_1–group exists that is not a B_2–group.

In Section 2.4 we will transfer the results of Eklof and Trlifaj for Ext to Bext and use them to construct groups A such that $\mathrm{Bext}(A, A) = 0$ but $\mathrm{Ext}(A, A) \neq 0$. Especially, we will construct a B_1–group A such that $\mathrm{Bext}(A, A) = 0$ but $\mathrm{Ext}(A, A) \neq 0$.

Chapter 3 is devoted to the cotorsion pairs for Bext, called B–cotorsion pairs. In analogy to Göbel, Shelah and Wallutis [GSW] we will show that every power set can be embedded into the lattice of all B–cotorsion pairs. Hence there exist ascending and descending chains as well as anti–chains of arbitrary length in the lattice of all B–cotorsion pairs. In Section 3.2 we investigate the B–cotorsion pairs cogenerated by a rational group. Bican and Fuchs [BF] have called a torsion–free group A an R–group if $\mathrm{Bext}(A, R) = 0$ for a rational group R. We will use their results on R–groups to show that all B–cotorsion pairs singly cogenerated by a rational group are incomparable. This led to the question which groups A satisfy $\mathrm{Ext}(A, R) = 0$ for a rational group R. Since the case $R = \mathbb{Z}$ yields the Whitehead groups, we call such groups A R–Whitehead groups. In 1952 Whitehead asked the famous question if every group A which satisfies $\mathrm{Ext}(A, \mathbb{Z}) = 0$ is free. Shelah [Sh1] showed in 1974 that Whitehead's problem is undecidable in ZFC. He proved that assuming $V = L$ all Whitehead groups of cardinality $< \aleph_{\omega_1}$ are free. In contrast to that he constructed a non–free Whitehead group of cardinality $\aleph_1$ assuming Martin's axiom and $2^{\aleph_0} > \aleph_1$. One year later Shelah [Sh2] showed that assuming $V = L$ every Whitehead group is free. In 1980 Shelah [Sh3] translated Whitehead's problem into a combinatorial problem. He proved that there is a non–free Whitehead group of cardinality $\aleph_1$ if and only if there is a ladder system on a stationary subset of ω_1 which satisfies 2–uniformization. The results for R–Whitehead groups are similar. If we assume $V = L$, then every torsion–free R–Whitehead group A is R_0–free, i.e. the R_0–module $A \otimes R_0$ is free (here R_0 denotes the nucleus of R). In Section 4.4 we show that there is a non–R_0–free R–Whitehead group of cardinality $\aleph_1$ if and only if there is a ladder system on a stationary subset of ω_1 which satisfies 2–uniformization. This shows that in any model of set theory there is a non–free Whitehead group of cardinality $\aleph_1$ if and only if there is a non–R_0–free R–Whitehead group of cardinality $\aleph_1$.

List of symbols

Let A and C be arbitrary abelian groups, $x \in A$, R a rational group, G a torsion–free group and $g \in G$. Moreover, let p be a prime, S a set, κ an ordinal, f a function, $\mathcal{U}$ a p–indicator, τ a type and M a height matrix.

ORD	class of all ordinals
LORD	class of all limit ordinals
CARD	class of all cardinals
ω	first infinite ordinal
ω_1	first uncountable ordinal
$\aleph_0$	first infinite cardinal
$\aleph_1$	first uncountable cardinal
$\mathrm{cf}(\kappa)$	cofinality of κ
$\lvert S\rvert$	cardinality of S
$\mathcal{P}(S)$	power set of S
$\widetilde{S}$	equivalence class of S
Π	set of all primes
Π_R	set of all primes which divide R
Π_0^R	$= \Pi\backslash\Pi_R$
R_0	largest idempotent type less than or equal to R (nucleus)
$R(p)$	rational group isomorphic to R with p does not divide 1 in $R(p)$
$f^{-1}[A]$	the pre–image of A under f
$\mathrm{rk}(G)$	rank of $G = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes G)$
$\mathrm{rk}_p(A)$	p–rank of A
$\langle x\rangle_G$	subgroup generated by x in G
$D(A)$	maximal divisible subgroup of A
A_r	complementary summand of $D(A)$ in A (not unique)
$\mathrm{T}(A)$	torsion part of A
$\mathrm{T_p}(A)$	p–torsion part of A
$A[n]$	all elements $a \in A$ such that $na = 0$
$\mathrm{h}_p^A(x)$	p–height of x in A
$\mathcal{U}_p^A(x)$	p–indicator of x in A
$\mathrm{H}^A(x)$	height–sequence of x in A
$\mathbb{H}^A(x)$	height–matrix of x in A
$\mathrm{t}^G(g)$	type of g in G

$\mathrm{t}(R)$	type of R = type of 1 in R
$\chi_G(g)$	characteristic of g in G
$A(\mathcal{U})$	all elements of A with p–indicator $\geq \mathcal{U}$
$A(M)$	all elements of A with height matrix $\geq M$
$A(\tau)$	all elements of A of type $\geq \tau$
$\mathrm{Hom}(C, A)$	group of homomorphisms from C to A
$\mathrm{Ext}(C, A)$	group of extensions of A by C
$\mathrm{Bext}(C, A)$	group of balanced extensions of A by C

1 Preliminaries

In the following let A, B and C be groups. All groups under consideration will be abelian. Denote the class of ordinals by ORD, the class of limit ordinals by LORD and the class of cardinals by CARD.
For a prime p and an ordinal σ we define the subgroup $p^\sigma A$ of A inductively.

$$p^0A = A,\ p^\sigma A = p(p^\alpha A) \text{ if } \sigma = \alpha + 1 \text{ and } p^\sigma A = \bigcap_{\alpha<\sigma} p^\alpha A \text{ if } \sigma \in \text{LORD}$$

For every $a \in A$ there is a unique ordinal σ such that $a \in p^\sigma A \backslash p^{\sigma+1}A$ or $a \in p^\alpha A$ for all $\alpha \in \text{ORD}$. In the first case let $\mathrm{h}_p^A(a) = \sigma$, in the second case $\mathrm{h}_p^A(a) = \infty$. $\mathrm{h}_p^A(a)$ is called the *p–height* of a in A. The sequence of p–heights

$$\mathrm{H}^A(a) = (\mathrm{h}_p^A(a))_{p\in\Pi}$$

is called *height sequence* of a in A. For torsion–free groups A it is also called the *characteristic* of a in A and denoted by $\chi_A(a)$.
The sequence $\mathcal{U}_p^A(a) = (\mathrm{h}_p^A(p^k a))_{k\in\omega}$ is called the *p–indicator* of a in A.
We define the *height matrix* $\mathbb{H}^A(a)$ of a in A as the matrix which first column is the height sequence of a and which rows are the corresponding p–indicators of a.
For a given p–indicator $\mathcal{U}$ and a given height matrix M we define the subgroups $A(\mathcal{U})$ and $A(M)$ of A as the subgroups which contain all elements a with $\mathcal{U}_p^A(a) \geq \mathcal{U}$ resp. $\mathbb{H}^A(a) \geq M$. Here the relation $\leq$ is defined componentwise.

1.1 Torsion–free groups of rank 1

For torsion–free groups A the p–height in A of an element $a \in A$ is a natural number or ∞. We say that two characteristics are equivalent if they differ only in finitely many finite entries. Such an equivalence class of characteristics is called *type*. If the characteristic of a belongs to the type $\mathbf{t}$, then we say that a is of type $\mathbf{t}$ in A and write $t^A(a) = \mathbf{t}$. In analogy to the above definitions we define $A(\mathbf{t})$ to be the set of all elements in A of type $\geq \mathbf{t}$, where the relation $\leq$ on types is induced by the relation $\leq$ on the characteristics. A torsion–free group A in which all the elements are of the same type is called *homogeneous*.
We know that every torsion–free group of rank 1 is isomorphic to a subgroup of the rationals. Therefore, these groups are called *rational groups*. We can associate a type $\mathbf{t}$ to every rank 1 group R, where $\mathbf{t}$ is the type of all non–zero elements in R. It is well–known that two rank 1 groups are isomorphic if and only if they are of the same type.
We call a type $\mathbf{t}$ *idempotent* if there exists a characteristic $c \in \mathbf{t}$ such that c has only 0

and ∞ as entries. For a rational group R we will denote the largest subgroup of R of idempotent type by R_0. In fact, $R_0 \cong \mathrm{End}(R)$. We will denote the set of all primes p which divide R by Π_R and its complement $\Pi \backslash \Pi_R$ by Π_0^R. We will skip the index R if the group is obvious from the context.
R_0 is a principal ideal domain (PID) and if C is a torsion–free group, then $C \otimes R_0$ is a torsion–free R_0–module.

Lemma 1.1.1. *Let B and C be torsion–free groups. Every homomorphism $\Phi : B \to C \otimes R_0$ extends uniquely to a homomorphism $\hat{\Phi} : B \otimes R_0 \to C \otimes R_0$.*

Proof.
There is a unique extension $\hat{\Phi} : B \otimes R_0 \to C \otimes \mathbb{Q}$. $\hat{\Phi}$ induces a map $\psi : (B \otimes R_0)/B \to (C \otimes \mathbb{Q})/(C \otimes R_0)$ where both groups are torsion. While $(B \otimes R_0)/B$ contains only non–trivial p–components for $p \in \Pi_R$, $(C \otimes \mathbb{Q})/(C \otimes R_0)$ contains no non–trivial p–component for $p \in \Pi_R$. Hence $\psi = 0$ and $(B \otimes R_0)\hat{\Phi} \subseteq C \otimes R_0$. $\square$

Lemma 1.1.2. *Let*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & C \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"] & C \otimes R_0 \end{tikzcd}$$

be a pullback diagram where X and C are torsion–free groups, Y is a torsion–free R_0–module and α is the canonical embedding of C into $C \otimes R_0$. Then $Y \cong X \otimes R_0$.

Proof.
Since X is constructed as a pullback, $X = \{(c, y) | c\alpha = y\beta\}$. We define $\varphi : X \otimes R_0 \to Y$ by $(c, y) \otimes r_0 \mapsto r_0 y$. Then φ is well–defined and a homomorphism. It is easy to check that φ is also an isomorphism. $\square$

Lemma 1.1.3. *Let C be a torsion–free group. Then* $\mathrm{Ext}_{\mathbb{Z}}(C, R) = 0$ *if and only if* $\mathrm{Ext}_{R_0}(C \otimes R_0, R) = 0$.

Proof.
First assume that $\mathrm{Ext}_{R_0}(C \otimes R_0, R) = 0$ and that the sequence

$$0 \to R \xrightarrow{\alpha} X \xrightarrow{\delta} C \to 0$$

is exact. Then the following diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_R"'] & X \arrow[r, "\delta"] \arrow[d, "\gamma"'] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r, "\alpha'"] & X \otimes R_0 \arrow[r, "\delta'"] & C \otimes R_0 \arrow[r] & 0
\end{tikzcd}$$

commutes. Here $\alpha' = \alpha \otimes \mathrm{id}_{R_0}$ and $\delta' = \delta \otimes \mathrm{id}_{R_0}$. Since $\mathrm{Ext}_{R_0}(C \otimes R_0, R) = 0$, there is a homomorphism $\psi : X \otimes R_0 \to R$ such that $\alpha'\psi = \mathrm{id}_R$. Hence, $\alpha\gamma\psi = \mathrm{id}_R$ and, because it was an arbitrary sequence, $\mathrm{Ext}_{\mathbb{Z}}(C, R) = 0$.
Now assume that $\mathrm{Ext}_{\mathbb{Z}}(C, R) = 0$ and

$$0 \to R \to Y \xrightarrow{\hat{\delta}} C \otimes R_0 \to 0$$

is an exact sequence of R_0–modules. Then the diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, "\mathrm{id}_R"'] & X \arrow[r, "\delta"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r] & Y \arrow[r, "\hat{\delta}"] & C \otimes R_0 \arrow[r] & 0
\end{tikzcd}$$

where X is constructed as pullback, commutes and both rows are exact. In Lemma 1.1.2 we have shown that $Y \cong X \otimes R_0$. Since $\mathrm{Ext}_{\mathbb{Z}}(C, R) = 0$, the upper row splits and there is a homomorphism $\Phi : C \to X \subseteq X \otimes R_0$ such that $\Phi\delta = \mathrm{id}_C$. By Lemma 1.1.1 Φ can be extended to $\hat{\Phi} : C \otimes R_0 \to X \otimes R_0 \cong Y$. Then $\hat{\Phi}\hat{\delta} = \mathrm{id}_{C \otimes R_0}$, i.e. the second row splits. Therefore, $\mathrm{Ext}_{R_0}(C \otimes R_0, R) = 0$. $\square$

Later we will often use the relationship between C as abelian group and $C \otimes R_0$ as R_0–module.

An important generalization of the rank 1 groups are the completely decomposable groups.

Definition 1.1.4. A torsion–free group C is called *completely decomposable* if it is a direct sum of rank 1 groups.

Lemma 1.1.5. *Direct summands of completely decomposable groups are completely decomposable.*

Proof. See [F2, Theorem 86.7].

If we replace direct summand by pure subgroup, this result turns out to be false. There are pure subgroups of completely decomposable groups which are not completely decomposable (for an example see [A, Example 4.1]). Therefore, we have the following definition which goes back to Butler [B].

Definition 1.1.6. A torsion–free group C of finite rank is called *Butler group* if it satisfies one of the following two equivalent conditions.

(a) C is a pure subgroup of a completely decomposable group of finite rank;

(b) C is an epimorphic image of a completely decomposable group of finite rank.

Lemma 1.1.7. *Let C be a homogeneous finite rank Butler group. Then C is completely decomposable.*

Proof. For the proof see [A, Corollary 4.5].

There are two possible generalizations of Butler groups of finite rank to groups of arbitrary rank. Bican and Salce [BS] have introduced them in 1983. For the first definition, the definition of B_1–groups, we need the functor Bext, which will be introduced in Chapter 2. For the sake of completeness we will, nevertheless, give the definition of B_1–groups here.

Definition 1.1.8. Let C be a torsion–free group of arbitrary rank.

(a) C is called a B_1*–group* if $\mathrm{Bext}(C, T) = 0$ for all torsion groups T.

(b) C is called a B_2*–group* if there is a continuous well–ordered ascending chain of pure subgroups

$$0 = C_0 < C_1 < \cdots < C_\kappa = C = \bigcup_{\alpha<\kappa} C_\alpha$$

such that for all $\alpha < \kappa$ the quotient $C_{\alpha+1}/C_\alpha$ is of rank 1 and $C_{\alpha+1} = B_\alpha + C_\alpha$ for some finite rank Butler group B_α.

Bican and Salce [BS] showed that every B_2–group is a B_1–group and that for countable groups C these two definitions are equivalent. The answer to the question if all B_1–groups of arbitrary rank are also B_2–groups depends on set theory. In [FM], Fuchs and Magidor proved that in Gödel's universe L every B_1–group is a B_2–group and hence both classes coincide. 2003 Shelah and Strüngmann [SS] constructed a model of set theory in which a B_1–group exists that is not a B_2–group.

1.2 Algebraically compact groups and their relatives

We will need the following important classes of abelian groups.

Definition 1.2.1. Let A be an abelian group.

(a) A is *cotorsion* if $\mathrm{Ext}(C, A) = 0$ for all torsion–free groups C.

(b) A is called *algebraically compact* if A is a direct summand in every group C that contains A as a pure subgroup.

(c) A torsion group T is called *torsion–complete* if T is the torsion part of an algebraically compact group.

Obviously, the algebraically compact groups are exactly the pure–injective groups, i.e. for every pure–exact sequence

$$0 \to B \overset{\alpha}{\to} X \to C \to 0$$

and every homomorphism $\varphi : B \to A$ with A algebraically compact there exists a homomorphism $\psi : X \to A$ such that $\alpha\psi = \varphi$.

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "\alpha"] \arrow[d, "\varphi"] & X \arrow[r] \arrow[dl, "\psi"'] & C \arrow[r] & 0 \\
 & A & & &
\end{tikzcd}$$

Moreover, every algebraically compact group is cotorsion.

Lemma 1.2.2. *Let B be a reduced group and A a subgroup of B.*

(a) *If B is cotorsion, then a subgroup A of B is cotorsion if and only if B/A is reduced.*

(b) *If B is algebraically compact and $\bigcap_{n<\omega} n(B/A) = 0$, then A is algebraically compact.*

Proof. See [F1, 54.(B)] and [F1, 39.2].

Lemma 1.2.3. *Let T be a torsion–complete group. Then $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$ is algebraically compact.*

Proof. This follows directly from [F1, Proposition 54.2] and [F2, Theorem 68.4].

Lemma 1.2.4. *Let C be a reduced cotorsion group and T its torsion part. Then there is a direct decomposition $C = A \oplus B$ where A is torsion–free and algebraically compact and $B \cong \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$.*

Proof. For the proof see [F1, Theorem 55.5].

This Lemma shows that every torsion–free cotorsion group B is algebraically compact. By Corollary 40.4 of [F1] B contains a direct summand isomorphic to the rationals $\mathbb{Q}$ or isomorphic to the p–adic integers J_p for some prime p. This leads to the following definition of cotorsion–free groups.

Definition 1.2.5. Let B be a torsion–free group.

(a) B is called *cotorsion–free* if B does not contain a copy of the rationals $\mathbb{Q}$ or of the p–adic integers J_p for any prime p.

(b) B is called *ultra–cotorsion–free* if any subgroup A of B with $|A| = |B|$ and B/A cotorsion–free, equals B.

1.3 Balanced subgroups

Fuchs [F2] defined balanced subgroups and balanced–exact sequences for abelian p–groups and torsion–free abelian groups. First we will concentrate on p–groups.

Definition 1.3.1. Let $A \subseteq B$ be p–groups.

(a) A is called a *nice* subgroup of B, if for every coset $b + A$ of B/A there exists some $a \in A$ such that $\mathrm{h}_p^B(a + b) = \mathrm{h}_p^{B/A}(b + A)$.

(b) A is *isotype* in B if

$$p^\sigma A = p^\sigma B \cap A$$

for every ordinal σ.

(c) We call A *balanced* in B if A is a nice and isotype subgroup of B.

An easy example for a balanced subgroup is a direct summand. For non–trivial examples see [F2, Chapter 80].

Obviously, we call an exact sequence of p–groups

$$0 \to A \xrightarrow{\alpha} B \to C \to 0$$

balanced if $A\alpha$ is balanced in B.
In the following we will always identify A with the subgroup $A\alpha$ of B.

Lemma 1.3.2. *Let*

$$0 \to A \to B \xrightarrow{\beta} C \to 0$$

be an exact sequence of p–groups.
Then A is a balanced subgroup of B if and only if $(p^\sigma B[p])\beta = p^\sigma C[p]$ for all ordinals σ.

Proof. See [F2, Lemma 80.2].

Later we will need the following Lemma on nice subgroups.

Lemma 1.3.3. *Let A and C be subgroups of a p–group B such that $C \subseteq A \subseteq B$. Then the following hold.*

(a) *If A is nice in B, then A/C is nice in B/C.*

(b) *If C is nice in B and A/C is nice in B/C, then A is nice in B.*

Proof. See [F2, Lemma 79.3].

We can also characterize balanced subgroups using short exact sequences.

Lemma 1.3.4. *Let $A \subseteq B$ be p–groups. A is balanced in B iff the exact sequence*

$$0 \to A \to B \xrightarrow{\beta} C \to 0 \qquad (1)$$

implies the exactness of

$$0 \to p^\sigma A \to p^\sigma B \xrightarrow{\beta'} p^\sigma C \to 0$$

for every ordinal σ.

Proof.
To prove this Lemma we will show that

(a) A is isotype in B iff (1) implies the exactness of

$$0 \to p^\sigma A \to p^\sigma B \xrightarrow{\beta'} p^\sigma C$$

(b) A is nice in B if and only if $p^\sigma(B/A) = (p^\sigma B + A)/A$ for every $\sigma \in$ ORD.

Obviously (b) is equivalent to the statement that the homomorphism $\beta' : p^\sigma B \to p^\sigma C$ induced by β is surjective.

Proof of (a). Assume that A is isotype in B. Clearly $p^\sigma A \subseteq p^\sigma B$. Hence it remains to show that $p^\sigma A$ is the kernel of the map $p^\sigma B \overset{\beta'}{\to} p^\sigma C$. Since β' is induced by β, its kernel is $A \cap p^\sigma B = p^\sigma A$.

For the other direction assume that

$$0 \to p^\sigma A \to p^\sigma B \overset{\beta'}{\to} p^\sigma C$$

is exact. Then $x \in p^\sigma A = \operatorname{Ker} \beta'$ if and only if $x \in p^\sigma B \cap \operatorname{Ker} \beta = p^\sigma B \cap A$ and hence $p^\sigma A = p^\sigma B \cap A$.

Proof of (b). For every subgroup A of B holds $p^\sigma(B/A) \supseteq (p^\sigma B + A)/A$, since $p^\sigma(B/A)$ contains all elements of height $\geq \sigma$, while $(p^\sigma B + A)/A$ is the image of $p^\sigma B$ in B/A and hence contains only elements of height $\geq \sigma$.

A is nice in B if and only if every coset in $p^\sigma(B/A)$ can be represented by an element of B that is of height $\geq \sigma$, i.e. for every $x + A \in p^\sigma(B/A)$ exists an $b \in p^\sigma B$ such that $b + A = x + A$. Hence A is nice in B if and only if $p^\sigma(B/A) = (p^\sigma B + A)/A$. $\square$

In the following we will denote by $D(B)$ the maximal divisible subgroup of B. Then $D(B)$ is a direct summand of B. Hence there is a complementary summand B_r such that $B = D(B) \oplus B_r$ and $B_r \cong B/D(B)$. Note that this complement is not unique. If A is a subgroup of B, then we will choose the complementary summands of $D(A)$ and $D(B)$ such that $A_r \subseteq B_r$.

Lemma 1.3.5. *Let*

$$0 \to A \to B \to C \to 0 \tag{2}$$

be a balanced–exact sequence of p–groups.

Then the sequence of the maximal divisible subgroups is exact, too.

$$0 \to \mathrm{D}(A) \to \mathrm{D}(B) \to \mathrm{D}(C) \to 0 \tag{3}$$

Hence (2) *is balanced–exact if and only if* (3) *and*

$$0 \to A_r \to B_r \to C_r \to 0 \tag{4}$$

are balanced–exact.

Proof.

Assume that (2) is balanced–exact. Then $\mathrm{D}(A) = \mathrm{D}(B) \cap A$ and for every element $c \in \mathrm{D}(C)$

there is an element $b \in B$ of infinite height which is mapped onto c. This shows that (3) is balanced–exact whenever (2) is balanced–exact. Hence the following diagram commutes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D(A) \arrow[r] \arrow[d] & D(B) \arrow[r] \arrow[d] & D(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_r \arrow[r] \arrow[d] & B_r \arrow[r] \arrow[d] & C_r \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and the exactness of the first two rows implies the exactness of the last row. It remains to show that (4) is balanced. First we show isotypeness. Let $x \in p^\sigma B_r \cap A_r \subseteq p^\sigma B \cap A = p^\sigma A$. Then $x \in p^\sigma A \cap A_r = p^\sigma A_r$ since A_r is balanced in A as a direct summand. To show that A_r is nice in B_r let $x \in C_r$. Since C_r is a direct summand of C, $\mathrm{h}_p^C(x) = \mathrm{h}_p^{C_r}(x)$. The middle row of the above diagram is balanced, hence there is some $b \in B$ such that $\mathrm{h}_p^B(b) = \mathrm{h}_p^C(x) = \mathrm{h}_p^{C_r}(x)$ and $\mathrm{h}_p^{B_r}(b) = \mathrm{h}_p^B(b) = \mathrm{h}_p^{C_r}(x)$, i.e. A_r is nice in B_r.
Now assume that (3) and (4) are balanced–exact. Then (1) is exact as direct sum of (3) and (4). Every element $b \in B$ has a unique representation $b = x + y$ such that $x \in D(B)$ and $y \in B_r$. Now let $b \in p^\sigma B \cap A$. Then $x \in p^\sigma D(B) \cap D(A) = p^\sigma D(A)$ and $y \in p^\sigma B_r \cap A_r = p^\sigma A_r$. Hence $b = x + y \in p^\sigma A$. In the same way one can show that A is nice in B. □

Definition 1.3.6. Let A be a p–group. Then the *σ–th Ulm–Kaplansky invariant* of A is the cardinal number

$$f_\sigma(A) = \mathrm{rk}_p((p^\sigma A)[p]/(p^{\sigma+1}A)[p]).$$

Lemma 1.3.7. *There exists a family of p–groups H_σ ($\sigma \in$ ORD) such that*

(a) $H_0 = 0$;

(b) $p^\sigma H_{\sigma+1}$ *is cyclic of order* p;

(c) $H_{\sigma+1}/p^\sigma H_{\sigma+1} \cong H_\sigma$;

(d) $H_\sigma = \bigoplus_{\rho<\sigma} H_\rho$ *if* σ *is a limit ordinal;*

(e) *every Ulm–Kaplansky invariant of* H_σ *is* $\leq |\sigma|$.

H_σ *is called the* generalized Prüfer group of length σ.

Proof. In [F2, below 81.5] these groups are constructed in an inductive process.

The following Lemma shows one of the important features of the generalized Prüfer groups that we will need later on.

Lemma 1.3.8. *Let A be a p–group and $a \in (p^\sigma A)[p^n]$. Then there is a homomorphism $\Phi : H_{\sigma+n} \to A$ such that $h\Phi = a$, where h is a generator of $p^\sigma H_{\sigma+n}$.*

Proof. See [F2, Lemma 81.7].

Definition 1.3.9. Let T be a p–group and

$$0 = N_0 < N_1 < \cdots < N_\kappa = T$$

be a well–ordered strictly ascending chain of subgroups of T. This chain is called a *nice composition series* for T if

(a) $N_0 = \{0\}$ and $N_\kappa = T$;

(b) each N_λ is a nice subgroup of T;

(c) $N_{\lambda+1}/N_\lambda$ is countable for every $\lambda < \kappa$;

(d) $N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha$ if λ is a limit ordinal.

Definition 1.3.10. Let T be a reduced p–group. T is called *totally projective* if

$$p^\sigma \operatorname{Ext}(T/p^\sigma T, G) = \{0\}$$

for all ordinals σ and groups G.

There is a direct connection between the last two definitions.

Lemma 1.3.11. *Let T be a reduced p–group. Then the following are equivalent.*

(a) *T is totally projective;*

(b) *T has the projective property relative to all balanced–exact sequences of p–groups;*

(c) *T is a summand of a direct sum of generalized Prüfer groups;*

(d) *T has a nice composition series.*

Proof. For the proof see [F2, Theorem 82.3].

This lemma and the preceding definitions naturally generalize to arbitrary torsion groups. For the proof of the following proposition we will need some results on totally projective groups and nice subgroups.

Lemma 1.3.12. *Let A, B be p–groups and $x \in A$. Then the following hold.*

(a) *Every cyclic subgroup $\langle x \rangle$ of A is nice in A.*

(b) *If A is totally projective, then $A/\langle x \rangle$ is totally projective, too.*

(c) *If C is a nice subgroup of A and A/C is totally projective, then every homomorphism $\Phi : C \to B$, which does not decrease heights, can be extended to a homomorphism from A to B.*

Proof. For the proof see [He, Lemma I.2.4, Lemma I.2.11, Corollary I.2.14].

Proposition 1.3.13. *Let*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

be a balanced–exact sequence of p–groups and $\mathcal{U} = (\tau_0, \tau_1, \dots)$ an increasing sequence of ordinals and ∞.
Then

$$0 \to A(\mathcal{U}) \to B(\mathcal{U}) \to C(\mathcal{U}) \to 0$$

is balanced–exact, too.

Proof.
Without loss of generality we may assume that the groups A, B and C are reduced by Lemma 1.3.5 since $G(\mathcal{U}) = G_r(\mathcal{U}) \oplus \mathrm{D}(G)$ for every group G and every p–indicator $\mathcal{U}$.
Now let $c \in C(\mathcal{U})$ and $\mathrm{o}(c) = n$. We want to construct a totally projective group T such that there exists $x \in T$ with $\mathcal{U}_p^T(x) = \mathcal{U}_p^C(c)$ and a homomorphism $\Phi : T \to C$ such that x

is mapped onto c. We distinguish the two cases if the p–indicator of c has a gap or not. If $\mathcal{U}_p^C(c) = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1}, \infty, \ldots)$ has no gap (besides the gap following σ_{n-1}), let $T = H_{\sigma_0+n}$ be the generalized Prüfer group of length $\sigma_0 + n$. Lemma 1.3.8 shows that there is a homomorphism $\Phi : T \to C$ such that the generator h of $p^{\sigma_0} H_{\sigma_0+n}$ is mapped onto c. Of course $\mathcal{U}_p^T(h) = \mathcal{U}_p^C(c)$ holds. Hence choose $x = h$.
If $\mathcal{U}_p^C(c) = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1}, \infty, \ldots)$ has more gaps than the trivial gap following σ_{n-1}, we define T as a direct sum of generalized Prüfer groups. Let $\sigma_{i_1}, \ldots, \sigma_{i_k} = \sigma_{n-1}$ be the k entries which are followed by a gap. We set $T_1 := H_{\sigma_0+i_1+1}$ and choose h_1 as a generator of $p^{\sigma_0} H_{\sigma_0+i_1+1}$. Then $\mathcal{U}_p^{T_1}(h_1) = (\sigma_0, \sigma_1, \ldots, \sigma_{i_1}, \infty, \ldots)$. If there exists an ordinal γ_2 such that $\gamma_2 + i_1 + 1 = \sigma_{i_1+1}$, let $T_2 = H_{\gamma_2+i_2+1}$ and h_2 a generator of $p^{\gamma_2} H_{\gamma_2+i_2+1}$. Then $\mathcal{U}_p^{T_2}(h_2) = (\gamma_2, \gamma_2 + 1, \ldots, \gamma_2 + i_1 + 1 = \sigma_{i_1+1}, \ldots, \sigma_{i_2}, \infty, \ldots)$. If there is no such ordinal, let $T_2 = H_{\sigma_{i_1+1}+(i_2-i_1)}$ and choose $\tilde{h}_2$ a generator of $p^{\sigma_{i_1+1}} H_{\sigma_{i_1+1}+(i_2-i_1)}$. Since $\sigma_{i_1+1} > \sigma_0 + i_1 + 1$, there exists $h_2 \in p^{\sigma_0} H_{\sigma_{i_1+1}+(i_2-i_1)}$ such that $p^{i_1+1} h_2 = \tilde{h}_2$. Then $\mathcal{U}_p^{T_2}(h_2) = (\alpha_0, \ldots, \alpha_{i_1}, \sigma_{i_1+1}, \ldots, \sigma_{i_2}, \infty, \ldots)$ with some ordinals $\alpha_0, \ldots, \alpha_{i_1}$. Since $\alpha_0 \geq \sigma_0$ and hence $\alpha_j \geq \sigma_j$ for all $j \leq i_1$, we have in both cases $\mathcal{U}_p^{T_1 \oplus T_2}(h_1 + h_2) = (\sigma_0, \ldots, \sigma_{i_1}, \sigma_{i_1+1}, \ldots, \sigma_{i_2}, \infty, \ldots)$. We can proceed in this way and define $T_1, \ldots, T_k$ and $h_1, \ldots, h_k$ such that $\mathcal{U}_p^T(x) = \mathcal{U}_p^C(c)$, where $T = \bigoplus_{j=1}^{k} T_j$ and $x = \sum_{j=1}^{k} h_j$. Of course, T is totally projective as a direct sum of generalized Prüfer groups. Moreover, there is a homomorphism φ from $\langle x \rangle$ to C such that x is mapped onto c. By Lemma 1.3.12 we can extend φ to a homomorphism $\Phi : T \to C$.
Now, we have the following situation.

$$\begin{tikzcd}
 & & & T \arrow[dl, "\psi"'] \arrow[d, "\phi"] & \\
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0
\end{tikzcd}$$

The map $\psi : T \to B$ exists because T is totally projective and hence has the projective property relative to all balanced–exact sequences of p–groups by Lemma 1.3.11.
Our x is mapped onto some $b = x\psi$ such that $b\beta = c$. On the one hand we have $\mathcal{U}_p^B(b) \geq \mathcal{U}_p^T(x) = \mathcal{U}_p^C(c)$, but on the other hand $\mathcal{U}_p^B(b) \leq \mathcal{U}_p^C(c)$ and hence $\mathcal{U}_p^B(b) = \mathcal{U}_p^C(c)$.
This construction can be applied to any $c \in C(\mathcal{U})$ and therefore, β restricted to $B(\mathcal{U})$ is an epimorphism onto $C(\mathcal{U})$. Moreover, the kernel of this epimorphism is nice in $B(\mathcal{U})$. It remains to show that $A(\mathcal{U}) = A \cap B(\mathcal{U})$ and that $A(\mathcal{U})$ is isotype in $B(\mathcal{U})$.
Of course, $A(\mathcal{U}) \subseteq A \cap B(\mathcal{U})$. Let $y \in A \cap B(\mathcal{U})$. Then $\mathcal{U}_p^B(y) \geq \mathcal{U} = (\tau_0, \tau_1, \ldots)$ and hence $y \in p^{\tau_0} B \cap A = p^{\tau_0} A$ because A is isotype in B. In the same way we have

$p^n y \in p^{\tau_n} B \cap A = p^{\tau_n} A$ for all $n \in \mathbb{N}$. Hence $\mathcal{U}_p^A(y) \geq \mathcal{U}$ and $y \in A(\mathcal{U})$. This implies that the sequence

$$0 \to A(\mathcal{U}) \to B(\mathcal{U}) \to C(\mathcal{U}) \to 0$$

is exact.
We use induction to show that $p^\sigma B(\mathcal{U}) \cap A(\mathcal{U}) = p^\sigma A(\mathcal{U})$. Let $\sigma = 1$ and $pb = x \in pB(\mathcal{U}) \cap A(\mathcal{U})$. Then there exists $a' \in A(\mathcal{U})$ such that $\mathcal{U}_p^{C(\mathcal{U})}(b + A(\mathcal{U})) = \mathcal{U}_p^{B(\mathcal{U})}(b - a') \geq \mathcal{U}_p^{B(\mathcal{U})}(b)$. Since the groups are reduced, $\mathrm{o}(b - a') = \mathrm{o}(b + A(\mathcal{U})) = p$ and therefore, $x = pb = pa' \in pA(\mathcal{U})$. The case where σ is a limit ordinal is clear, so assume $\sigma = \gamma + 1$. Let $x \in p^\sigma B(\mathcal{U}) \cap A(\mathcal{U})$. Then there exists $y \in p^\gamma B(\mathcal{U})$ such that $py = x$. We have seen above that there is $a' \in A(\mathcal{U})$ with $\mathcal{U}_p^{C(\mathcal{U})}(y + A(\mathcal{U})) = \mathcal{U}_p^{B(\mathcal{U})}(y - a') \geq \mathcal{U}_p^{B(\mathcal{U})}(y)$. The last inequality implies $\mathcal{U}_p^{B(\mathcal{U})}(a') \geq \mathcal{U}_p^{B(\mathcal{U})}(y)$ and hence $a' \in p^\gamma B(\mathcal{U}) \cap A(\mathcal{U}) = p^\gamma A(\mathcal{U})$ by the induction hypothesis. Again all our groups are reduced, whence we have $\mathrm{o}(y - a') = \mathrm{o}(y + A(\mathcal{U})) = p$. Alltogether, $x = py = pa' \in p^{\gamma+1} A(\mathcal{U})$ and we have shown that $A(\mathcal{U})$ is balanced in $B(\mathcal{U})$. □

In accordance to the definition of balanced subgroups for p–groups, Fuchs [F2] defined a subgroup A of a torsion–free group B to be balanced if A is pure in B and for every coset $b + A$ of B/A with $b \notin A$ there exists some $a \in A$ such that $\chi_B(b + a) = \chi_{B/A}(b + A)$.
He also characterized the torsion–free balanced–projectives.

Lemma 1.3.14. *Completely decomposable groups have the projective property relative to all balanced–exact sequences of torsion–free groups.*

Proof. See [F2, Theorem 86.2].

Hunter [Hu] generalized the definition of balanced subgroups to arbitrary abelian groups. With the help of Lemma 1.3.4 and Proposition 1.3.13 one can easily see that his definition extends the definitions by Fuchs.

Definition 1.3.15. A short–exact sequence

$$0 \to A \xrightarrow{\alpha} B \to C \to 0$$

is called *balanced* if the sequence

$$0 \to A(M) \to B(M) \to C(M) \to 0$$

is exact for every height–matrix M. In this case we will also say that $A\alpha$ is balanced in B.

Remark 1.3.16. *One can easily see that Lemma 1.3.5 holds also for arbitrary groups. If*

$$0 \to A \to B \to C \to 0$$

is a balanced–exact sequence, then the sequence

$$0 \to \mathrm{D}(A) \to \mathrm{D}(B) \to \mathrm{D}(C) \to 0$$

of maximal divisible subgroups is exact and hence splits off, i.e. we can always assume without loss of generality that A, B and C are reduced.

The next definition generalizes the notion of nice and isotype subgroups.

Definition 1.3.17. Let $A \subseteq B$.

(a) A is called *H–nice* in B if for every coset $b + A$ of B/A with $b \notin A$ there exists some $a \in A$ such that $\mathbb{H}^B(a + b) = \mathbb{H}^{B/A}(b + A)$. The element $a + b$ is called *proper* with respect to A.

(b) A is *p–isotype* in B if

$$p^\sigma A = p^\sigma B \cap A$$

for every ordinal σ.

(c) A is called *isotype* in B if A is p–isotype in B for every prime p.

We will call a subgroup A of B *p–nice* if it satisfies the condition of Definition 1.3.1(a), i.e. if for every coset $b + A$ of B/A with $b \notin A$ there exists some $a \in A$ such that $\mathrm{h}_p^B(a + b) = \mathrm{h}_p^{B/A}(b + A)$. Then $a + b$ is called *p–proper* with respect to A. In the same way we define A to be a *p–balanced* subgroup of B if A is p–isotype and p–nice in B.

Remark 1.3.18. *Note that if $A \subseteq B$ are p–groups and A is nice in B, A is not necessarily H–nice in B.*
For example let $B = \mathbb{Z}/9\mathbb{Z}$ and $A = 3B$. Then A and B are 3–groups, $A \subseteq B$ and $B/A \cong \mathbb{Z}/3\mathbb{Z}$. Moreover, A is a cyclic subgroup of B and by Lemma 1.3.12(a) nice in B. To see that A is not H–nice in B, consider the element $1 + 3\mathbb{Z} \in B/A$. It is

$$\mathrm{h}_3^{B/A}(1 + 3\mathbb{Z}) = 0 \quad \textit{and} \quad \mathrm{h}_3^{B/A}(3(1 + 3\mathbb{Z})) = \infty.$$

The inverse image of $1 + 3\mathbb{Z}$ in B is the set $\{1 + 9\mathbb{Z}, 4 + 9\mathbb{Z}, 7 + 9\mathbb{Z}\}$. We have

$$\mathrm{h}_3^B(1 + 9\mathbb{Z}) = 0 \quad \textit{and} \quad \mathrm{h}_3^B(3(1 + 9\mathbb{Z})) = 1,$$

$$\mathrm{h}_3^B(4+9\mathbb{Z}) = 0 \quad and \quad \mathrm{h}_3^B(3(4+9\mathbb{Z})) = 1,$$

$$\mathrm{h}_3^B(7+9\mathbb{Z}) = 0 \quad and \quad \mathrm{h}_3^B(3(7+9\mathbb{Z})) = 1.$$

Hence there is no element in the inverse image of $1+3\mathbb{Z}$ *which has the same* 3*–indicator as* $1+3\mathbb{Z}$*, i.e.* A *is not H–nice in* B*.*
Proposition 1.3.13 *shows that if a subgroup of a p–group is nice and isotype, then it is also H–nice.*

Lemma 1.3.19. *Let* $A \subseteq B$ *and* $\mathrm{T_p}(B/A) = 0$*. Then* A *is p–isotype in* B*.*

Proof.
We will show by induction that $p^\sigma B \cap A = p^\sigma A$ for all $\sigma \in \mathrm{ORD}$. Let $\sigma = 1$ and $pb = a \in pB \cap A$. Then $p(b+A) = 0$ and since B/A has no p–torsion, $b + A = 0$. Hence $b \in A$ and $a \in pA$. If σ is a limit ordinal, there is nothing to show. Now assume that $\sigma = \alpha+1$. If $a \in p^\sigma B \cap A$, we can write $a = pb$ with $b \in p^\alpha B$. Then $p(b+A) = 0$ and hence $b+A = 0$. It is $b \in A \cap p^\alpha B = p^\alpha A$ by induction hypothesis and therefore $a = pb \in p^\sigma A$. $\square$

This lemma implies directly

Corollary 1.3.20. *The torsion part* $\mathrm{T}(B)$ *is always isotype in* B*.*

There are several equivalent characterizations of balanced subgroups.

Lemma 1.3.21. *Let* $A \subseteq B$*,* $B/A = C$ *and* $\beta : B \to C$ *be the natural epimorphism. Then the following are equivalent.*

(a) *A is balanced in B;*

(b) *A is isotype and H–nice in B;*

(c) *for every* $c \in C$ *there is some* $b \in B$ *such that* $b\beta = c$*,* $\mathbb{H}^B(b) = \mathbb{H}^C(c)$ *and* $\mathrm{o}(b) = \mathrm{o}(c)$*;*

(d) $0 \to A/A(M) \to B/B(M) \to C/C(M) \to 0$ *is exact for every height–matrix* M*.*

Proof.
$(a) \Leftrightarrow (b)$. First we will show that $\beta(M) : B(M) \to C(M)$ is surjective for every height–matrix M if and only if A is H–nice in B. Assume that $\beta(M)$ is surjective for every height–matrix M and let $b+A \in B/A = C$. Let $N = \mathbb{H}^{B/A}(b+A)$. Then $b+A \in C(N)$ and since $\beta(N)$ is surjective, there exists $b' \in B(N)$ such that $b'+A = b+A$ and $\mathbb{H}^B(b') \geq N$. On the other hand $\mathbb{H}^B(b') \leq \mathbb{H}^{B/A}(b'+A) = N$ and hence A is H–nice in B.
Now assume that A is H–nice in B and let $b+A \in C(M)$ for some height–matrix M.

Then $\mathbb{H}^C(b+A) \geq M$ and there exists $b' \in B$ such that $b'+A = b+A$ and $\mathbb{H}^B(b') = \mathbb{H}^C(b+A) \geq M$. Hence $b' \in B(M)$ and $\beta(M)$ is surjective.

It remains to show that $\mathrm{Ker}(\beta(M)) = A(M)$ for all height–matrices M if and only if A is isotype in B. Obviously, $\mathrm{Ker}(\beta(M)) = B(M) \cap A$ for all M. It is $B(M) \cap A = A(M)$ iff $\mathbb{H}^B(a) \geq M$ implies $\mathbb{H}^A(a) \geq M$ for every $a \in A$. Let σ_{pk}, p prime, $k \geq 0$ denote the entries of M. Then this is equivalent to $\mathrm{h}_p^B(p^k a) \geq \sigma_{pk}$ implies $\mathrm{h}_p^A(p^k a) \geq \sigma_{pk}$, i.e. $p^k a \in p^{\sigma_{pk}} B \cap A$ if and only if $p^k a \in p^{\sigma_{pk}} A$. Hence $B(M) \cap A = A(M)$ for all M is equivalent to $p^\sigma A = p^\sigma B \cap A$ for all ordinals σ, i.e. A is isotype in B.

$(a) \Leftrightarrow (d)$. The commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A(M) \arrow[r] \arrow[d] & B(M) \arrow[r] \arrow[d] & C(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A/A(M) \arrow[r] \arrow[d] & B/B(M) \arrow[r] \arrow[d] & C/C(M) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

has exact columns and exact middle row. Hence by the 3x3–lemma the first row is exact if and only if the last row is exact.

$(a) \Rightarrow (c)$. By Remark 1.3.16 we can assume that A, B and C are reduced since (c) is trivially fulfilled for the sequence of divisible subgroups. We have already shown that if A is balanced in B, then A is H–nice in B. Hence for every $c \in C$ there exists some $b \in B$ such that $b\beta = b + A = c$ and $\mathbb{H}^B(b) = \mathbb{H}^C(c)$. In a reduced group the order of an element is uniquely determined by its height–matrix and therefore $\mathrm{o}(c) = \mathrm{o}(b)$.

$(c) \Rightarrow (b)$. We only have to show that A is isotype in B. We do this by induction on $\sigma \in \mathrm{ORD}$. If $\sigma = 0$, there is nothing to show. If σ is a limit ordinal,

$$p^\sigma A = \bigcap_{\alpha<\sigma} p^\alpha A = \bigcap_{\alpha<\sigma} (p^\alpha B \cap A) = \bigcap_{\alpha<\sigma} p^\alpha B \cap A = p^\sigma B \cap A.$$

Now let $\sigma = \alpha+1$ and let $x \in p^\sigma B \cap A$. Then there is $y \in p^\alpha B$ such that $py = x$. If y is also an element of A, we are done. Hence assume $y \notin A$. Then $\mathrm{o}(y+A) = p$. By assumption there is $z \in B$ such that $z + A = y + A$, $\mathrm{o}(z) = p$ and $\mathbb{H}^B(z) = \mathbb{H}^C(y+A)$, especially $\mathrm{h}_p^B(z) \geq \alpha$ and $z \in p^\alpha B$. Therefore, $y - z \in A \cap p^\alpha B = p^\alpha A$ and $p(y-z) = py = x$, i.e. $x \in p^{\alpha+1}A = p^\sigma A$. □

There is the following interesting corollary.

Corollary 1.3.22. *If the group B is reduced, then a subgroup A is balanced in B if and only if A is H–nice in B.*

We will need the following basic results on balanced subgroups.

Lemma 1.3.23. *Let $A \subseteq B \subseteq C$ and B balanced in C. Then B/A is balanced in C/A.*

Proof.
We have to show that B/A is H–nice and isotype in C/A. First let $c + B \in C/B \cong (C/A)/(B/A)$. Since B is balanced in C, there is some $b \in B$ such that $\mathbb{H}^C(c+b) = \mathbb{H}^{C/B}(c+B)$. Hence $\mathbb{H}^{C/A}(c+b+A) \geq \mathbb{H}^C(c+b) = \mathbb{H}^{C/B}(c+B)$ and B/A is H–nice in C/A.
To show isotypeness, let $x + A \in p^\sigma(C/A) \cap B/A$. Then there is some $a \in A$ such that $x + a \in p^\sigma C \cap B = p^\sigma B$ because B is isotype in C. Hence $x + A \in (p^\sigma B + A)/A \subseteq p^\sigma(B/A)$. □

Lemma 1.3.24. *Let $A \subseteq B \subseteq C$, A balanced in B and B balanced in C. Then A is balanced in C.*

Proof.
A is isotype in C since

$$p^\sigma A = A \cap p^\sigma B = A \cap B \cap p^\sigma C = A \cap p^\sigma C.$$

To show H–niceness, let $c + A \in C/A$ and $\mathbb{H}^{C/A}(c+A) = (\sigma_{pk})_{p,k}$. We distinguish the two cases if c is in B or not. First let $c \in B$. Then $c + A \in B/A$ and B/A is balanced in C/A by Lemma 1.3.23. Hence

$$p^k(c+A) \in p^{\sigma_{pk}}(C/A) \cap B/A = p^{\sigma_{pk}}(B/A)$$

for all p, k. Therefore, $\mathbb{H}^{B/A}(c+A) = (\sigma_{pk})_{p,k}$. Since A is balanced in B, there is $a \in A$ such that $\mathbb{H}^B(c+a) = (\sigma_{pk})_{p,k}$. Then

$$\mathbb{H}^C(c+a) \geq \mathbb{H}^B(c+a) = \mathbb{H}^{C/A}(c+A)$$

and equality follows immediately.
Now assume that $c \notin B$. Then $\mathbb{H}^{C/B}(c+B) \geq (\sigma_{pk})_{p,k}$. Since B is balanced in C, there is some $b \in B$ such that $\mathbb{H}^{C/B}(c+B) = \mathbb{H}^{C}(c+b)$ and since B/A is balanced in C/A, by Lemma 1.3.23 we have

$$\mathbb{H}^{C/A}(-b+A) = \mathbb{H}^{B/A}(-b+A) = (\sigma_{pk})_{p,k}.$$

Moreover, since A is balanced in B, there is $a \in A$ such that

$$\mathbb{H}^{B/A}(-b+A) = \mathbb{H}^{B}(-b+a) \leq \mathbb{H}^{C}(-b+a).$$

Hence

$$\mathbb{H}^{C}(a+c) = \mathbb{H}^{C}(-b+a+b+c) \geq (\sigma_{pk})_{p,k}$$

and in both cases A is H–nice in C. □

Lemma 1.3.25. *Let I be an index set and $A_i \subseteq B_i$ for all $i \in I$. Then $\bigoplus_{i\in I} A_i$ is balanced in $\bigoplus_{i\in I} B_i$ if and only if A_i is balanced in B_i for every $i \in I$.*

Proof.
This follows directly from the fact that $x = (x_i)_{i\in I} \in p^{\sigma}\bigoplus_{i\in I} B_i$ if and only if $x_i \in p^{\sigma}B_i$ for every $i \in I$. □

Lemma 1.3.26. *Let $(A_\alpha)_{\alpha<\kappa}$ be a continuous ascending chain of abelian groups such that A_α is balanced in $A_{\alpha+1}$ for all $\alpha < \kappa$. Then A_α is balanced in A_β for all $\alpha < \beta < \kappa$.*

Proof.
Let $\alpha < \kappa$. We will prove the Lemma by induction on β. By assumption A_α is balanced in $A_{\alpha+1}$. Now assume that A_α is balanced in A_γ for all $\alpha < \gamma < \beta$.
If $\beta = \tau + 1$, then A_α is balanced in A_τ and A_τ is balanced in A_β. Hence A_α is balanced in A_β by Lemma 1.3.24.
Now let β be a limit ordinal. First we show that A_α is isotype in A_β. Let $x \in p^{\sigma}A_\alpha \backslash p^{\sigma+1}A_\alpha$. Since A_α is isotype in A_γ for all $\alpha < \gamma < \beta$, we also have $x \in p^{\sigma}A_\gamma \backslash p^{\sigma+1}A_\gamma$ for all $\gamma < \beta$. Then $x \in p^{\sigma}A_\beta \backslash p^{\sigma+1}A_\beta$ and hence $p^{\sigma}A_\alpha = p^{\sigma}A_\beta \cap A_\alpha$ for all σ.
It remains to show that A_α is H–nice in A_β. Let $x + A_\alpha \in A_\beta/A_\alpha = \bigcup_{\alpha<\gamma<\beta} A_\gamma/A_\alpha$ and $\alpha < \alpha_0 = \tau + 1 < \beta$ such that $x \in A_{\alpha_0} \backslash A_\tau$. A_{α_0}/A_α is balanced and in particular isotype in A_γ/A_α for all $\alpha_0 < \gamma < \beta$ by Lemma 1.3.23. Hence $\mathrm{h}_p^{A_{\alpha_0}/A_\alpha}(x + A_\alpha) =$

$\mathrm{h}_p^{A_\gamma/A_\alpha}(x+A_\alpha)$ for all $\alpha_0<\gamma<\beta$. Therefore, $\mathrm{h}_p^{A_{\alpha_0}/A_\alpha}(x+A_\alpha)=\mathrm{h}_p^{A_\beta/A_\alpha}(x+A_\alpha)$ and $\mathbb{H}^{A_{\alpha_0}/A_\alpha}(x+A_\alpha)=\mathbb{H}^{A_\beta/A_\alpha}(x+A_\alpha)$. Since A_α is balanced in A_{α_0}, there is $\tilde{x}\in A_{\alpha_0}\subseteq A_\beta$ such that $\mathbb{H}^{A_{\alpha_0}}(\tilde{x})=\mathbb{H}^{A_{\alpha_0}/A_\alpha}(x+A_\alpha)=\mathbb{H}^{A_\beta/A_\alpha}(x+A_\alpha)$ and $\tilde{x}+A_\alpha=x+A_\alpha$. Hence $\mathbb{H}^{A_\beta}(\tilde{x})=\mathbb{H}^{A_\beta/A_\alpha}(x+A_\alpha)$ and A_α is H–nice in A_β. □

1.4 Basics of set theory

In this section we will recall the most basic set–theoretical notions, definitions and results which we will need later. Recall that ORD denotes the class of all ordinals, LORD the class of all limit ordinals and CARD the class of all cardinals.

Definition 1.4.1. Let κ be a limit ordinal and C and S subsets of κ.

(a) C is called *closed* in κ if for all $Y\subseteq C$ with $\sup Y\in\kappa$ also $\sup Y\in C$.

(b) C is called *unbounded* in κ if $\sup C=\kappa$.

(c) C is called a *cub* (in κ) if C is closed and unbounded in κ.

(d) S is called *stationary* in κ if $S\cap C\neq\emptyset$ for every cub C.

We can define an equivalence relation on all subsets of κ. We say that S and S' are equivalent if there is a cub C in κ such that $S\cap C=S'\cap C$ and denote the equivalence class of S by $\widetilde{S}$.

Lemma 1.4.2. *Let κ be a limit ordinal and $\lambda<\mathrm{cf}(\kappa)$. If $\bigcup\{X_\nu:\nu<\lambda\}$ is stationary in κ, then there exists $\nu<\lambda$ such that X_ν is stationary in κ.*

Proof. See [EM, Corollary II.4.5].

Lemma 1.4.3. *Let κ be a regular uncountable cardinal. Then every stationary subset E of κ can be partitioned into κ disjoint stationary sets.*

Proof. For the proof see [J, Theorem 85].

Let A be a set of cardinality $\leq\kappa$. Then we call an indexed sequence $\{A_\nu:\nu<\kappa\}$ a *κ–filtration* of A if for all $\nu<\kappa$

(a) $|A_\nu|<\kappa$;

(b) for all $\mu < \nu$ is $A_\mu \subseteq A_\nu$;

(c) if $\nu \in \mathrm{LORD}$, then $A_\nu = \bigcup\limits_{\mu<\nu} A_\mu$;

(d) $A = \bigcup\limits_{\nu<\kappa} A_\nu$.

If $\{A'_\nu : \nu < \kappa\}$ is another κ–filtration of A, then $C = \{\nu \in \kappa : A_\nu = A'_\nu\}$ is a cub in κ (see [EM, II.4.12]).

Definition 1.4.4. Let R be a principal ideal domain (PID) and M an R–module. Then M is called *κ–free* if every submodule of cardinality $< \kappa$ is free.

The following lemma, called Pontryagin's criterion, is well–known.

Lemma 1.4.5. *Let R be a PID and M an R–module. Then M is $\aleph_1$–free iff every finite rank submodule of M is free.*

Proof. For the proof see [EM, Theorem IV.2.3].

Naturally, the question arises, when a κ–free module is free. The following singular compactness theorem treats the case of singular cardinals κ. It was first proved by Shelah in 1975 (see [Sh2]).

Lemma 1.4.6. *Let κ be a singular cardinal and M a $\leq \kappa$–generated module which is κ–free. Then M is free.*

Proof. For the proof see [EM, Theorem IV.3.5].

In the case of regular cardinals κ the Γ–invariant of a κ–free module M is a useful tool to decide whether M is free or not.

Definition 1.4.7. Let κ be a regular cardinal and M a κ–free, $\leq \kappa$–generated module over a PID. Moreover, let $\{M_\nu : \nu < \kappa\}$ be a κ–filtration of M. Let

$$E = \{\nu < \kappa : \exists\ \mu > \nu \text{ such that } M_\mu/M_\nu \text{ is not free}\}$$

or, equivalently, $E = \{\nu < \kappa : M/M_\nu \text{ is not } \kappa\text{–free}\}$.
Then we call $\Gamma(M) = \widetilde{E}$ the Γ*–invariant* of M.

In [EM, below Definition IV.1.6] it is shown that the Γ–invariant is really an invariant and does not depend on the choice of the filtration of M.

Lemma 1.4.8. *Let κ be a regular uncountable cardinal and M a $\leq \kappa$–generated κ–free module. Then M is free if and only if $\Gamma(M) = 0$.*

Proof. For the proof see [EM, Proposition IV.1.7].

2 The functor Bext

In this chapter we will investigate the functor Bext. In Section 1.3 we have recalled the definition of balanced subgroups of p–groups and torsion–free groups as given by Fuchs [F2] and its generalization for arbitrary groups by Hunter [Hu]. The balanced–exact sequences form a proper class in the sense of Mac Lane [M]. They define a subfunctor Bext of Ext and hence the full machinery of the relative homological algebra is available. Bican and Salce [BS] have shown that the functor Bext plays an important role in the theory of Butler groups. They proved that a finite rank torsion–free group G is a Butler group if and only if $\mathrm{Bext}(G, T) = 0$ for all torsion groups T. This led to the definition of B_1–groups as given in Definition 1.1.8(a).

2.1 Balanced–exact sequences

Let E be a balanced extension of A by C. Then for all homomorphisms $\alpha : A \to A'$ and $\gamma : C' \to C$ the extensions αE and $E\gamma$ are again balanced. Especially, every extension E' of A by C which is equivalent to E is balanced. Hence the extensions E of A by C which are represented by balanced–exact sequences form a subgroup of $\mathrm{Ext}(C, A)$. We will denote this subgroup by $\mathrm{Bext}^1(C, A)$ and call it the *group of balanced extensions* of A by C. Then $\mathrm{Bext}^1(\_\,,\_)$ is a subfunctor of $\mathrm{Ext}(\_\,,\_)$.
Moreover, it follows that for a balanced–exact sequence

$$0 \to A \to B \to C \to 0$$

the sequences

$$0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G) \to$$

$$\mathrm{Bext}^1(C, G) \to \mathrm{Bext}^1(B, G) \to \mathrm{Bext}^1(A, G)$$

and

$$0 \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, B) \to \mathrm{Hom}(G, C) \to$$

$$\mathrm{Bext}^1(G, A) \to \mathrm{Bext}^1(G, B) \to \mathrm{Bext}^1(G, C)$$

are exact, too.
In general these two long–exact sequences will continue with $\mathrm{Bext}^2(C, G)$ resp. $\mathrm{Bext}^2(G, A)$ since these groups are not necessarily equal to zero as it is the case for $\mathrm{Ext}^2(\_\,,\_)$. In the following we will only consider the group $\mathrm{Bext}^1(\_\,,\_)$ and therefore skip the index 1.

Remark 2.1.1. *By Remark 1.3.16*

$$\mathrm{Bext}(C, A) \cong \mathrm{Bext}(C_r, A_r),$$

i.e. we can always assume without loss of generality that C and A are reduced.

Lemma 2.1.2. *If the sequence*

$$0 \to A \to B \xrightarrow{\beta} C \to 0$$

is balanced–exact, then

$$0 \to \mathrm{T}(A) \to \mathrm{T}(B) \xrightarrow{\beta} \mathrm{T}(C) \to 0$$

is balanced–exact, too.

Proof.
First we show that the sequence of the torsion parts is exact. Since A is balanced in B, for every $c \in \mathrm{T}(C)$ there is $b \in B$ such that $b\beta = c$ and $\mathrm{o}(b) = \mathrm{o}(c)$ by Lemma 1.3.21. Hence $b \in \mathrm{T}(B)$ and the sequence is exact at $\mathrm{T}(C)$. The exactness at $\mathrm{T}(B)$ follows from $\mathrm{T}(B) \cap A = \mathrm{T}(A)$.
It remains to show that $\mathrm{T}(A)$ is balanced in $\mathrm{T}(B)$. Lemma 1.3.2 shows that this holds if and only if $(p^\sigma \mathrm{T}(B)[p])\beta = p^\sigma \mathrm{T}(C)[p]$ for all primes p and ordinals σ. The inclusion "$\subseteq$" is obvious. Hence let $x \in p^\sigma \mathrm{T}(C)[p]$ and choose $b \in p^\sigma B$ such that $b\beta = x$. Then $pb \in p^{\sigma+1}B \cap A = p^{\sigma+1}A$ and $pb = pa$ for some $a \in p^\sigma A$. Since $\mathrm{T}(B)$ is isotype in B by Corollary 1.3.20, it follows that $b - a \in p^\sigma \mathrm{T}(B)[p]$ and $(b-a)\beta = b\beta = x$. Therefore, $\mathrm{T}(A)$ is balanced in $\mathrm{T}(B)$. $\square$

With the help of the 3×3–lemma we get the following

Corollary 2.1.3. *Let C be a torsion group. Then* $\mathrm{Bext}(C, \mathrm{T}(A)) = 0$ *implies* $\mathrm{Bext}(C, A) = 0$.

The question arises, which properties of Ext carry over to Bext. Two very useful properties of Ext are

$$\mathrm{Ext}(\bigoplus_{i\in I} C_i, A) \cong \prod_{i\in I} \mathrm{Ext}(C_i, A)$$

and

$$\mathrm{Ext}(C, \prod_{i\in I} A_i) \cong \prod_{i\in I} \mathrm{Ext}(C, A_i)$$

for arbitrary groups $A, C, A_i (i \in I), C_i (i \in I)$ and index set I.
The first isomorphism also holds for Bext.

Lemma 2.1.4. *Let I be an index set and $A, C_i (i \in I)$ arbitrary groups. Then*

$$\operatorname{Bext}(\bigoplus_{i\in I} C_i, A) \cong \prod_{i\in I} \operatorname{Bext}(C_i, A).$$

Proof.
In Theorem 2.3.8 we will show that every group is a balanced epimorphic image of a balanced–projective group. Hence there are balanced–exact sequences

$$0 \to K_i \to B_i \to C_i \to 0$$

where B_i is balanced–projective for every $i \in I$. By Lemma 1.3.25 the sequence

$$0 \to \bigoplus_{i\in I} K_i \to \bigoplus_{i\in I} B_i \to \bigoplus_{i\in I} C_i \to 0$$

is balanced–exact. Hence the diagram

$$\begin{tikzcd}
\operatorname{Hom}(\bigoplus_{i\in I} B_i, A) \arrow[r] \arrow[d, "\cong"'] & \operatorname{Hom}(\bigoplus_{i\in I} K_i, A) \arrow[r] \arrow[d, "\cong"'] & \operatorname{Bext}(\bigoplus_{i\in I} C_i, A) \to 0 \\
\prod_{i\in I} \operatorname{Hom}(B_i, A) \arrow[r] & \prod_{i\in I} \operatorname{Hom}(K_i, A) \arrow[r] & \prod_{i\in I} \operatorname{Bext}(C_i, A) \to 0
\end{tikzcd}$$

commutes and therefore

$$\operatorname{Bext}(\bigoplus_{i\in I} C_i, A) \cong \prod_{i\in I} \operatorname{Bext}(C_i, A)$$

holds. □

Corollary 2.1.5. *If C and A are torsion groups, then*

$$\operatorname{Bext}(C, A) \cong \prod_{p\in\Pi} \operatorname{Bext}(\mathrm{T_p}(C), \mathrm{T_p}(A)).$$

Proof.
If C and A are torsion, then we have by Lemma 2.1.4

$$\operatorname{Bext}(C, A) = \operatorname{Bext}(\bigoplus_{p\in\Pi} \mathrm{T_p}(C), \bigoplus_{q\in\Pi} \mathrm{T_q}(A)) \cong \prod_{p\in\Pi} \operatorname{Bext}(\mathrm{T_p}(C), \bigoplus_{q\in\Pi} \mathrm{T_q}(A)).$$

The isomorphism shows that we can restrict ourselves to p–groups C. Then the sequence

$$0 \to \bigoplus_{q\in\Pi} \mathrm{T_q}(A) \to X \to C \to 0$$

is balanced–exact if and only if $\bigoplus_{q\in\Pi\setminus\{p\}} \mathrm{T_q}(X) \cong \bigoplus_{q\in\Pi\setminus\{p\}} \mathrm{T_q}(A)$ and

$$0 \to \mathrm{T_p}(A) \to \mathrm{T_p}(X) \to C \to 0$$

is balanced–exact. Hence $\mathrm{Bext}(C, \bigoplus_{q\in\Pi} \mathrm{T_q}(A)) \cong \mathrm{Bext}(C, \mathrm{T_p}(A))$ for every p–group C. $\square$

The next proposition shows that the second of the above isomorphisms for Ext carries over to Bext if C and $A_i (i \in I)$ are torsion–free. For arbitrary abelian groups we have a weaker result.

Proposition 2.1.6. *Let I be an index set and $C, A_i (i \in I)$ arbitrary groups. Then*

$$\mathrm{Bext}(C, \prod_{i\in I} A_i) \subseteq \prod_{i\in I} \mathrm{Bext}(C, A_i).$$

If C and $A_i (i \in I)$ are torsion–free, we even have

$$\mathrm{Bext}(C, \prod_{i\in I} A_i) \cong \prod_{i\in I} \mathrm{Bext}(C, A_i).$$

Proof.
Without loss of generality we can assume that $A_i (i \in I)$ and C are reduced. This will make the calculation a little bit easier.
We know that there is an isomorphism $\psi : \mathrm{Ext}(C, \prod_{i\in I} A_i) \to \prod_{i\in I} \mathrm{Ext}(C, A_i)$. Since $\mathrm{Bext}(C, \prod_{i\in I} A_i)$ is a subgroup of $\mathrm{Ext}(C, \prod_{i\in I} A_i)$, it is enough to show that every balanced–exact sequence in $\mathrm{Ext}(C, \prod_{i\in I} A_i)$ is mapped onto a balanced–exact sequence. Therefore, we need to have a closer look at the isomorphism ψ. For every i let D_i be the divisible hull of A_i. Then

$$\widetilde{E}_i : 0 \to A_i \to D_i \to D_i/A_i \to 0$$

and

$$\widetilde{E} : 0 \to \prod_{i\in I} A_i \to \prod_{i\in I} D_i \to \prod_{i\in I} D_i/A_i \to 0$$

are exact (but not balanced).
Moreover, we know that the following diagram, where ϕ and ψ are isomorphisms, commutes.

$$\begin{tikzcd}
\mathrm{Hom}(C, \prod_{i\in I} D_i/A_i) \arrow[r, "\widetilde{E}_*"] \arrow[d, "\phi"] & \mathrm{Ext}(C, \prod_{i\in I} A_i) \to 0 \arrow[d, "\psi"] \\
\prod_{i\in I} \mathrm{Hom}(C, D_i/A_i) \arrow[r, "\prod \widetilde{E}_{i*}"] & \prod_{i\in I} \mathrm{Ext}(C, A_i) \to 0
\end{tikzcd}$$

Hence for every sequence E in $\mathrm{Ext}(C, \prod_{i\in I} A_i)$ there is a homomorphism $\alpha : C \to \prod_{i\in I} D_i/A_i$ such that $E = \widetilde{E}\alpha$.

$$\begin{tikzcd}
E : 0 \to \prod_{i\in I} A_i \arrow[r] \arrow[d, "\mathrm{id}"'] & X \arrow[r, "\gamma"] \arrow[d, "\delta"'] & C \to 0 \arrow[d, "\alpha"'] \\
\widetilde{E} : 0 \to \prod_{i\in I} A_i \arrow[r] & \prod_{i\in I} D_i \arrow[r, "\beta"] & \prod_{i\in I} D_i/A_i \to 0
\end{tikzcd}$$

Here X is constructed as pullback, i.e.

$$X = \{(c,d) | c \in C, d \in \prod_{i\in I} D_i, c\alpha = d\beta\}$$

and γ and δ are the projections on the first, resp. second coordinate.
Since ϕ and ψ are isomorphisms, the following diagram with exact rows commutes for every $i \in I$.

$$\begin{tikzcd}
E_i : 0 \to A_i \arrow[r] \arrow[d, "\mathrm{id}"'] & X_i \arrow[r, "\gamma_i"] \arrow[d, "\delta_i"'] & C \to 0 \arrow[d, "\alpha\pi_i"'] \\
\widetilde{E}_i : 0 \to A_i \arrow[r] & D_i \arrow[r, "\beta_i"] & D_i/A_i \to 0
\end{tikzcd}$$

Here π_i denotes the projection onto the i–th coordinate. Again X_i is constructed as pullback, i.e.

$$X_i = \{(c,d_i) | c \in C, d_i \in D_i, c\alpha\pi_i = d_i\beta_i\}$$

and γ_i and δ_i are the projections on the first, resp. second coordinate. Moreover, $d\beta = (d_i\beta_i)_{i\in I}$.
We show that if E is balanced, then E_i is balanced for all $i \in I$. This means that every balanced–exact sequence in $\mathrm{Ext}(C, \prod_{i\in I} A_i)$ is mapped onto a balanced–exact sequence by ψ. Therefore, assume that $\prod_{i\in I} A_i$ is balanced in X. Since we have assumed that A_i and C are reduced, it is enough to show that A_i is H–nice in X_i. Let $c \in C$. Then there is $(c,d) \in X$ such that

$$\mathbb{H}^C(c) = \mathbb{H}^X((c,d)) \le \mathbb{H}^{X_i}((c,d_i)) \le \mathbb{H}^C(c)$$

for every $i \in I$. Hence A_i is balanced in X_i for all $i \in I$ and we have shown that $\mathrm{Bext}(C, \prod_{i\in I} A_i) \subseteq \prod_{i\in I} \mathrm{Bext}(C, A_i)$.
Now assume that C and A_i are torsion–free for all $i \in I$. We will show that if A_i is

balanced in X_i for all $i \in I$, then $\prod_{i\in I} A_i$ is balanced in X. Therefore, ψ restricted to Bext is an isomorphism.

Let A_i be balanced in X_i for all $i \in I$. Again we only have to show that $\prod_{i\in I} A_i$ is H–nice in X. Let $c \in C$. Then there exists $(c, d_i) \in X_i$ with $\mathbb{H}^{X_i}((c, d_i)) = \mathbb{H}^C(c)$ for all $i \in I$ because A_i is H–nice in X_i for all $i \in I$. Let $d = (d_i)_{i\in I}$. We show that $\mathbb{H}^X((c, d)) = \mathbb{H}^{X_i}((c, d_i)) = \mathbb{H}^C(c)$. Let $\mathrm{h}_p^C(c) = n$ for some $n \in \mathbb{N} \cup \{\infty\}$. First assume that $n \in \mathbb{N}$. Since C, A_i and hence X_i are torsion–free, there is exactly one $(\tilde{c}, \tilde{d}_i) \in X_i$ such that $p^n\tilde{c} = c$ and $p^n\tilde{d}_i = d_i$. Moreover, we have $\tilde{c}\alpha\pi_i = \tilde{d}_i\beta_i$. Let $\tilde{d} = (\tilde{d}_i)_{i\in I}$. Then $(\tilde{c}, \tilde{d}) \in X$ and $p^n(\tilde{c}, \tilde{d}) = (c, d)$. A similar calculation shows that we get the same result for $n = \infty$. Hence $\mathbb{H}^X(c, d) = \mathbb{H}^C(c)$ and therefore, $\prod_{i\in I} A_i$ is H–nice in X. Thus $\mathrm{Bext}(C, \prod_{i\in I} A_i) \cong \prod_{i\in I} \mathrm{Bext}(C, A_i)$ for torsion–free groups $C, A_i (i \in I)$. $\square$

Question. Does the isomorphism

$$\mathrm{Bext}(C, \prod_{i\in I} A_i) \cong \prod_{i\in I} \mathrm{Bext}(C, A_i)$$

also hold for arbitrary groups $C, A_i (i \in I)$ or are there groups $C, A_i (i \in I)$ such that $\mathrm{Bext}(C, \prod_{i\in I} A_i)$ is properly contained (via ψ) in $\prod_{i\in I} \mathrm{Bext}(C, A_i)$?

2.2 Balanced–injective groups

We call a group A *balanced–injective* if it has the injective property relative to all balanced–exact sequences of abelian groups, i.e. if for every balanced–exact sequence

$$0 \to B \xrightarrow{\alpha} X \to C \to 0$$

and every homomorphism $\varphi : B \to A$ there exists a homomorphism $\psi : X \to A$ such that $\alpha\psi = \varphi$.

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "\alpha"] \arrow[d, "\varphi"] & X \arrow[r] \arrow[dl, "\psi"] & C \arrow[r] & 0 \\
& A & & &
\end{tikzcd}$$

Hunter [Hu] has characterized the groups A which have the injective property relative to all balanced–exact sequences of abelian groups. In this section we will follow his proof and show that the balanced–injective groups are exactly the pure–injective groups.

First we will consider sequences of torsion–free groups.

Lemma 2.2.1. *Let C be a torsion–free, homogeneous group of type $\mathbb{Z}$. Then every short–exact sequence*

$$0 \to A \to B \xrightarrow{\beta} C \to 0$$

is balanced.

Proof.
Since $B/A \cong C$ is torsion–free, Lemma 1.3.19 implies that A is isotype in B. By Lemma 1.3.21 it remains to show that A is H–nice in B. Let $c \in C$. C is homogeneous of type $\mathbb{Z}$ and hence there exist $c' \in C$ and $n \in \mathbb{N}$ such that $nc' = c$ and $\chi(c') = (0, \dots, 0, \dots)$. For every $b \in B$ with $b\beta = c'$ we have $nb\beta = c$ and hence $\chi(nb) = \chi(c)$. Moreover, $p^k c = p^k nc'$ implies that A is H–nice in B. $\square$

Now, we can characterize the groups which have the injective property relative to all balanced–exact sequences of torsion–free groups.

Proposition 2.2.2. *A group A has the injective property relative to all balanced–exact sequences of torsion–free groups if and only if A is cotorsion.*

Proof.
If A is cotorsion, then $\mathrm{Ext}(W, A) = 0$ and hence $\mathrm{Bext}(W, A) = 0$ for all torsion–free W. Now assume that A has the injective property relative to all balanced–exact sequences of torsion–free groups. Choose a cardinal $\kappa \geq |A|$ such that $\kappa^{\aleph_0} = 2^\kappa$. Note that such a cardinal always exists, for example choose $\kappa = \lim\{2^{|A|}, 2^{2^{|A|}}, \dots\}$. Let $X = \prod_\kappa \mathbb{Z}$ and $Y = \bigoplus_\kappa \mathbb{Z}$. Then $|X| = \aleph_0^\kappa = 2^\kappa$ and $|Y| = \kappa$. We denote the divisible part of X/Y by D and the pre–image of D under the natural epimorphism from X onto X/Y by W. Note that W is $\mathbb{Z}$–homogeneous. Let $\{p_k : k \in \omega\}$ be an enumeration of the set of all primes. Take a countable subset $\{a_i : i \in \omega\}$ of κ and let $\prod_{j=1}^{i} p_j^i$ be the entry in the a_i–th coordinate of an element x of X/Y. Then $x \in D$. There are $\kappa^{\aleph_0}$ almost disjoint countable subsets of κ, so we have $|D| = \kappa^{\aleph_0} = 2^\kappa$ and $|W| = 2^\kappa$.
Let

$$0 \to U \to V \to W \to 0$$

be a free presentation of W. We have shown in Lemma 2.2.1 that this sequence is balanced–exact since W is $\mathbb{Z}$–homogeneous. Moreover,

$$\cdots \to \mathrm{Hom}(V, A) \xrightarrow{\Phi} \mathrm{Hom}(U, A) \to \mathrm{Ext}(W, A) \to \mathrm{Ext}(V, A) = 0$$

and

$$\cdots \to \mathrm{Hom}(V, A) \xrightarrow{\Phi} \mathrm{Hom}(U, A) \to \mathrm{Bext}(W, A) = 0$$

are exact. Hence Φ is an epimorphism and $\mathrm{Ext}(W, A) = 0$.
Assume that A is not cotorsion. Then $\mathrm{Ext}(\mathbb{Q}, A) \neq 0$. The exactness of the sequence

$$0 \to Y \to W \to \bigoplus_{2^\kappa} \mathbb{Q} \to 0$$

implies the exactness of

$$\cdots \to \mathrm{Hom}(Y, A) \xrightarrow{\nu} \mathrm{Ext}(\bigoplus_{2^\kappa} \mathbb{Q}, A) \to \mathrm{Ext}(W, A) = 0.$$

On the one hand $|\,\mathrm{Hom}(Y, A)| \leq \prod_\kappa |A| \leq \kappa^\kappa = 2^\kappa$, on the other hand $|\,\mathrm{Ext}(\bigoplus_{2^\kappa} \mathbb{Q}, A)| = \prod_{2^\kappa} |\,\mathrm{Ext}(\mathbb{Q}, A)| \geq 2^{2^\kappa}$. This contradicts the fact that ν is an epimorphism. Hence A has to be cotorsion. □

The corresponding result for torsion groups was shown by Griffith.

Lemma 2.2.3. *A reduced p–group G has the injective property relative to all balanced–exact sequences of p–groups iff G is torsion–complete.*

Proof. See [G, Theorem 3.6].

Now, we can characterize the balanced–injective groups.

Theorem 2.2.4. *A group A is balanced–injective if and only if A is pure–injective.*

Proof.
Since balanced subgroups are pure, every pure–injective group is balanced–injective. Let A be balanced–injective. Without loss of generality we may assume that A is reduced. By Proposition 2.2.2 A has to be cotorsion. Obviously $\mathrm{T}(A)$ is injective with respect to all balanced–exact sequences of torsion groups. By Lemma 2.2.3 $\mathrm{T}(A)$ has to be torsion–complete and by Lemma 1.2.4 there is a direct decomposition $A = G \oplus H$ where G is algebraically compact and $H \cong \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, \mathrm{T}(A))$. Then by Lemma 1.2.3 H and hence A is algebraically compact, i.e. pure–injective. □

There are not enough balanced–injectives, i.e. not every group can be embedded as a balanced subgroup into a balanced–injective group. This follows directly from the next lemma.

Lemma 2.2.5. *Every balanced subgroup of a balanced–injective group is itself balanced–injective and hence a direct summand.*

Proof.
Let A be a balanced subgroup of a balanced–injective group B. Without loss of generality we may assume that A and B are reduced. Since B is balanced–injective, B is algebraically compact by Theorem 2.2.4. Hence $\bigcap\limits_{n<\omega} nB = 0$. A is balanced and especially H–nice in B, so we also have $\bigcap\limits_{n<\omega} n(B/A) = 0$. By Lemma 1.2.2(b) A is algebraically compact and therefore, balanced–injective. $\square$

2.3 Balanced–projective groups

We call a group A *balanced–projective* if it has the projective property relative to all balanced–exact sequences of abelian groups, i.e. if for every balanced–exact sequence

$$0 \to B \to X \xrightarrow{\beta} C \to 0$$

and every homomorphism $\varphi : A \to C$ there exists a homomorphism $\psi : A \to X$ such that $\psi\beta = \varphi$.

$$\begin{tikzcd}
& & & A \arrow[dl, "\psi"'] \arrow[d, "\varphi"] & \\
0 \arrow[r] & B \arrow[r] & X \arrow[r, "\beta"] & C \arrow[r] & 0
\end{tikzcd}$$

Hunter [Hu] also characterized the balanced–projective groups. In this section we will only show the most interesting steps of his proof.
Fuchs has shown in [F2, 81.9] that the balanced–projective groups relative to short–exact sequences of torsion groups are exactly the totally projective groups. For the general case we have to study groups A of torsion–free rank at most 1. Therefore, let $a \in A$ be an element of infinite order and T a torsion group such that

$$0 \to \langle a \rangle \to A \to T \to 0$$

is exact.
For a subgroup B of A and a prime p we define

$$A(p, B) := \{a \in A : p^k a \in B \text{ for some } k \geq 0\}.$$

Lemma 2.3.1. *Let A have torsion–free rank 1 and $a \in A$ be an element of infinite order. Then $\langle a \rangle$ is p–nice in $A(p, \langle a \rangle)$.*

Proof.
Let $z + \langle a \rangle \in A(p, \langle a \rangle)/\langle a \rangle$ and $\mathrm{h}_p(z) = \sigma$. Every element of the coset $z + \langle a \rangle$ can be written as $z + ka$ with $k \in \mathbb{Z}$. If $\mathrm{h}_p(ka) \neq \mathrm{h}_p(z)$ for all $k \in \mathbb{Z}$, then $\mathrm{h}_p(z + ka) = \min\{\mathrm{h}_p(z), \mathrm{h}_p(ka)\} \leq \sigma$ for all $k \in \mathbb{Z}$. Moreover, $\sigma = \mathrm{h}_p(z) \leq \mathrm{h}_p(z + \langle a \rangle)$ and hence $\mathrm{h}_p(z + \langle a \rangle) = \sigma = \mathrm{h}_p(z)$, i.e. z is p–proper with respect to $\langle a \rangle$.
Now assume that $\mathrm{h}_p(z) = \sigma = \mathrm{h}_p(p^n a)$ for some $n \in \mathbb{N}_0$. We show that the set $E = \{\mathrm{h}_p(z + kp^n a) : (k, p) = 1\}$ is finite and hence contains a maximal element. If $\mathrm{h}_p(z + k_1 p^n a) < \mathrm{h}_p(z + k_2 p^n a)$ for $k_1, k_2 \in \mathbb{Z}$ and $(k_1, p) = 1 = (k_2, p)$, then

$$\mathrm{h}_p(z + k_1 p^n a) = \mathrm{h}_p(z + k_1 p^n a - (z + k_2 p^n a)) = \mathrm{h}_p((k_1 - k_2)p^n a) = \mathrm{h}_p(p^m a)$$

for some $m \geq n$. Since $A(p, \langle a \rangle)/\langle a \rangle$ is p–torsion, there exists a positive integer r such that $p^r z = cp^l a$ and $(c, p) = 1$. We distinguish the two cases if $l = n + r$ or not.
First assume $l \neq n + r$. Then

$$\begin{aligned} \mathrm{h}_p(z + kp^n a) \leq \mathrm{h}_p(p^r z + kp^{n+r} a) &= \mathrm{h}_p(cp^l a + kp^{n+r} a) \\ &= \min\{\mathrm{h}_p(p^l a), \mathrm{h}_p(p^{n+r} a)\}. \end{aligned}$$

If $\mathrm{h}_p(z + kp^n a)$ is not maximal in E, we have shown before that

$$\mathrm{h}_p(z + kp^n a) = \mathrm{h}_p(p^m a) \leq \min\{\mathrm{h}_p(p^l a), \mathrm{h}_p(p^{n+r} a)\}$$

and hence $n \leq m \leq \min\{l, n + r\}$. This implies that there can be no strictly increasing chain in E. Therefore, there exists some $d \in \mathbb{Z}, (d, p) = 1$ such that $\mathrm{h}_p(z + dp^n a)$ is maximal in E, i.e. $z + dp^n a$ is p–proper with respect to $\langle a \rangle$.
Now assume that $l = n + r$. Then $p^r z = cp^l a = cp^{n+r} a$ implies $p^r(z - cp^n a) = 0$. Since z and $z - cp^n a$ are elements of the same coset, we can assume without loss of generality that $p^r z = 0$. Hence

$$\mathrm{h}_p(z + kp^n a) \leq \mathrm{h}_p(p^r z + kp^{n+r} a) = \mathrm{h}_p(kp^{n+r} a) = \mathrm{h}_p(p^{n+r} a).$$

Now we can argue as in the first case and whence E contains a maximal element. This maximal element is p–proper with respect to $\langle a \rangle$ and hence $\langle a \rangle$ is p–nice in $A(p, \langle a \rangle)$. □

It is easy to see that $A(p, \langle a \rangle)$ is the complete inverse image of $\mathrm{T_p}(A/\langle a \rangle)$ under the natural epimorphism.

Definition 2.3.2. Let $\mathcal{A}$ denote the class of groups A such that A is an extension of a cyclic group (finite or infinite) by a totally projective group.
Let $\mathcal{A}^\Sigma$ be the class of all direct sums of groups in $\mathcal{A}$ and denote the class of all direct summands of groups in $\mathcal{A}^\Sigma$ by $\bar{\mathcal{A}}$.

The class $\mathcal{A}$ contains all torsion–free groups of rank 1 and all totally projective groups. Moreover, we have

Lemma 2.3.3. *A torsion group is an element of $\mathcal{A}$ if and only if it is totally projective.*

Proof.
Obviously, a totally projective torsion group is an element of $\mathcal{A}$. Hence assume G to be a torsion group in $\mathcal{A}$. Without loss of generality we can assume that G is a p–group. Then G is an extension of a finite cyclic p–group $\langle g\rangle$ by a totally projective p–group T.

$$0 \to \langle g\rangle \to G \to T \to 0$$

We will show that G has a nice composition series and hence is totally projective. Let $N_\lambda, \lambda < \mu$ denote a nice composition series for T. Define M_λ to be the complete inverse image of N_λ in G. Then M_λ is a subgroup of G for all $\lambda < \mu$ and $M_0 = \langle g\rangle$ and $M_\mu = G$. Of course, for λ a limit ordinal we have $M_\lambda = \bigcup_{\kappa<\lambda} M_\kappa$. It is $G/\langle g\rangle \cong T$ and $M_\lambda/\langle g\rangle \cong N_\lambda$. Since N_λ is a nice subgroup of P, $M_\lambda/\langle g\rangle$ is a nice subgroup of $G/\langle g\rangle$. By 1.3.3(b) M_λ is a nice subgroup of G for every $\lambda < \mu$. It remains to show that $M_{\lambda+1}/M_\lambda$ is countable. But this is obvious because $M_{\lambda+1}/M_\lambda \cong (M_{\lambda+1}/\langle g\rangle)/(M_\lambda/\langle g\rangle) \cong N_{\lambda+1}/N_\lambda$, which is countable. Together with the (finite) nice composition series for $\langle g\rangle$ we finally have a nice composition series for G. □

Let A, N and V be groups and β a monomorphism from N to A (i.e. we can identify N with the subgruop $N\beta$ of A). We say that a homomorphism $\psi : N \to V$ *does not decrease heights in* A, if $\mathrm{H}^A(n\beta) \le \mathrm{H}^V(n\psi)$ for all $n \in N$ (remember that $\mathrm{H}^A(a)$ denotes the height sequence of a in A).

Lemma 2.3.4. *Let*

$$\begin{tikzcd}
 & 0 \arrow[r] & N \arrow[r] \arrow[d, "\psi"] & A \arrow[d, "\phi"] & \\
0 \arrow[r] & U \arrow[r] & V \arrow[r, "\alpha"] & W \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram where both rows are exact and U is p–balanced in V. Moreover, suppose that ψ does not decrease heights in A and that $a \in A$ is a p–proper element with respect to N such that $pa \in N$.
Then ψ can be extended to a map $\psi^* : \langle N, a\rangle \to V$ such that $a\psi^*\alpha = a\phi$ and ψ^* does not decrease heights.

Proof.
Let $\mathrm{h}_p^A(a) = \sigma$. Since U is p–nice in V, there is $v \in p^\sigma V$ such that $v\alpha = a\phi$. It is $pv - (pa)\psi \in \operatorname{Ker}\alpha$ and hence $pv - (pa)\psi \in U \cap p^{\sigma+1}V = p^{\sigma+1}U$. There is $u \in p^\sigma U$ such that $pu = pv - (pa)\psi$. Define $a\psi^* = v - u$ and for $x \in N$ let $x\psi^* = x\psi$. Then $\psi^* : \langle N, a\rangle \to V$ such that $a\psi^*\alpha = a\phi$. We first have to show that ψ^* is well–defined. Assume that $x = ka \in N \cap \langle a\rangle$, where k has to be divisible by p since otherwise a would be an element of N. Let $x = pk'a$. Then

$$x\psi^* = k'(pa)\psi^* = k'(pv - pu) = k'(pa)\psi = x\psi.$$

It remains to show that ψ^* does not decrease heights. If $y \in N$, clearly

$$\mathrm{H}^A(y) \leq \mathrm{H}^V(y\psi) = \mathrm{H}^V(y\psi^*)$$

since ψ does not decrease heights. Now let $y \in N$ and consider $\mathrm{H}^A(y + a)$. If $q \neq p$, we have

$$\mathrm{h}_q^A(y + a) = \mathrm{h}_q^A(py + pa) \leq \mathrm{h}_q^V((py + pa)\psi) = \mathrm{h}_q^V((py + pa)\psi^*) = \mathrm{h}_q^V((y + a)\psi^*).$$

Moreover, it is $\mathrm{h}_p^A(a) = \sigma \leq \mathrm{h}_p^V(v - u) = \mathrm{h}_p^V(a\psi^*)$. Since a is p–proper with respect to N, $\mathrm{h}_p^A(y + a) = \min\{\mathrm{h}_p^A(y), \mathrm{h}_p^A(a)\}$. Therefore,

$$\begin{aligned}\mathrm{h}_p^V((y + a)\psi^*) &= \mathrm{h}_p^V(y\psi + v - u) \geq \min\{\mathrm{h}_p^V(y\psi), \mathrm{h}_p^V(v - u)\}\\ &\geq \min\{\mathrm{h}_p^A(y), \sigma\} = \min\{\mathrm{h}_p^A(y), \mathrm{h}_p^A(a)\} = \mathrm{h}_p^A(y + a).\end{aligned}$$

It remains to consider $\mathrm{H}^A(y+ka)$ with $k > 1$ and p does not divide k. Since $\mathrm{h}_p^A(a) = \mathrm{h}_p^A(ka)$, we have $\mathrm{h}_p^A(y + ka) \leq \mathrm{h}_p^V((y + ka)\psi^*)$. If $q \neq p$, it follows again that

$$\mathrm{h}_q^A(y + ka) = \mathrm{h}_q^A(py + pka)$$

where $py + pka \in N$. Hence $\mathrm{H}^A(x) \leq \mathrm{H}^V(x\psi^*)$ for all $x \in \langle N, a\rangle$. □

The following lemma will be needed to construct the splitting map in the proof of the next theorem.

Lemma 2.3.5. *Let A and B be groups and H and $A_i (i \in I)$ subgroups of A such that $A = \sum_{i\in I} A_i$ and $A_i \cap \sum_{j\in I, j\neq i} A_j = H$ for all $i \in I$. Moreover, let $\phi_i : A_i \to B$ be homomorphisms such that $\phi_i\restriction H = \phi_j\restriction H$ for all $i, j \in I$. Then there exists a homomorphism $\Phi : A \to B$ such that $\Phi\restriction A_i = \phi_i$ for all $i \in I$.*

Proof. Clear!

Theorem 2.3.6. *If $A \in \bar{\mathcal{A}}$, then A is balanced–projective.*

Proof.
It is enough to consider groups $A \in \mathcal{A}$ since $\mathrm{Bext}(A_1 \oplus A_2, G) = \mathrm{Bext}(A_1, G) \oplus \mathrm{Bext}(A_2, G)$.
Let $A \in \mathcal{A}$, $\Phi : A \to W$ be a homomorphism and

$$0 \to U \to V \xrightarrow{\alpha} W \to 0$$

be a balanced–exact sequence. We will show that Φ lifts to a homomorphism $\Psi : A \to V$ such that $\Psi\alpha = \Phi$. First assume that A is torsion. We have shown in Lemma 2.3.3 that A is totally projective. Moreover, the sequence

$$0 \to \mathrm{T}(U) \to \mathrm{T}(V) \to \mathrm{T}(W) \to 0$$

is balanced–exact by Lemma 2.1.2 and Φ is a map from A into $\mathrm{T}(W)$. Lemma 1.3.11 shows that A has the projective property relative to this sequence and hence there exists a homomorphism $\Psi : A \to \mathrm{T}(V) \subseteq V$ such that $\Psi\alpha = \Phi$.
Now assume that A is of torsion–free rank 1. Let $a \in A$ with $\mathrm{o}(a) = \infty$ and choose $v \in V$ such that $v\alpha = a\Phi$ and $\mathbb{H}^W(a\Phi) = \mathbb{H}^V(v)$. Define a homomorphism $\psi' : \langle a \rangle \to V$ by $a \mapsto v$. Then ψ' does not decrease heights in A.
Next, we focus on the subgroups $A(p, \langle a \rangle)$ of A. In Lemma 2.3.1 we have shown that $\langle a \rangle$ is p–nice in $A(p, \langle a \rangle)$. Moreover, $\mathrm{T_p}(A/\langle a \rangle)$ is totally projective. Let

$$0 = N_0 < N_1 < \cdots < N_\mu = \mathrm{T_p}(A/\langle a \rangle)$$

be a nice composition series for $\mathrm{T_p}(A/\langle a \rangle)$, i.e. N_λ is p–nice in $\mathrm{T_p}(A/\langle a \rangle)$ for all $\lambda \le \mu$, $N_\lambda = \bigcup_{\kappa<\lambda} N_\kappa$ if $\lambda \in$ LORD and $N_{\lambda+1}/N_\lambda$ is countable for all $\lambda < \mu$. Define M_λ as the complete inverse image of N_λ in $A(p, \langle a \rangle)$. Then $M_0 = \langle a \rangle$ and $M_\mu = A(p, \langle a \rangle)$. Moreover, the sequence

$$0 \to \langle a \rangle \to M_\lambda \to N_\lambda \to 0$$

is exact for all $\lambda \le \mu$ and hence the diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \langle a\rangle \arrow[r] \arrow[d] & M_\lambda \arrow[r] \arrow[d] & N_\lambda \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \langle a\rangle \arrow[r] \arrow[d] & M_{\lambda+1} \arrow[r] \arrow[d] & N_{\lambda+1} \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & M_{\lambda+1}/M_\lambda \arrow[r] \arrow[d] & N_{\lambda+1}/N_\lambda \arrow[r] \arrow[d] & 0 \\
& & 0 & 0 &
\end{tikzcd}$$

commutes for all $\lambda < \mu$. By the 3×3–lemma the third row is exact and $M_{\lambda+1}/M_\lambda \cong N_{\lambda+1}/N_\lambda$. Whence $M_{\lambda+1}/M_\lambda$ is cyclic of order p and generated by every $0 \neq m_{\lambda+1} \in M_{\lambda+1} \backslash M_\lambda$. Moreover, $\langle a\rangle$ is p–nice in $A(p, \langle a\rangle)$ and $M_\lambda/\langle a\rangle \cong N_\lambda \underset{\text{p-nice}}{\subseteq} \mathrm{T_p}(A/\langle a\rangle) \cong A(p, \langle a\rangle)/\langle a\rangle$. Hence for every $\lambda \leq \mu$ there is $m_{\lambda+1} \in M_{\lambda+1} \backslash M_\lambda$ such that $m_{\lambda+1}$ is p–proper with respect to $A(p, \langle a\rangle)$.

Now let Φ_p denote the restriction of Φ to $A(p, \langle a\rangle)$ for $p \in \Pi$. Per transfinite induction we want to construct a homomorphism $\psi_p : A(p, \langle a\rangle) \to V$ such that $\psi_p \alpha = \Phi_p$. Therefore, consider for $\lambda \leq \mu$

$$\begin{tikzcd}
& 0 \arrow[r] & M_\lambda \arrow[r] \arrow[d, "\psi^p_\lambda"] & A(p, \langle a\rangle) \arrow[d, "\Phi_p"] & \\
0 \arrow[r] & U \arrow[r] & V \arrow[r, "\alpha"] & W \arrow[r] & 0
\end{tikzcd}$$

Let $\psi^p_0 = \psi' : \langle a\rangle \to V$ as defined above. ψ^p_0 does not decrease heights in $A(p, \langle a\rangle)$. Now assume that we have defined $\psi^p_\lambda : M_\lambda \to V$ such that the diagram above commutes and ψ^p_λ does not decrease heights $A(p, \langle a\rangle)$. Choose $m_{\lambda+1} \in M_{\lambda+1} \backslash M_\lambda \subseteq A(p, \langle a\rangle)$ such that $m_{\lambda+1}$ is p–proper with respect to $A(p, \langle a\rangle)$. Then $pm_{\lambda+1} \in M_\lambda$ and $\langle M_\lambda, m_{\lambda+1}\rangle = M_{\lambda+1}$. By Lemma 2.3.4 there is $\psi^p_{\lambda+1} : M_{\lambda+1} \to V$ such that $\psi^p_{\lambda+1}\alpha = \Phi_p \restriction M_{\lambda+1}$ and $\psi^p_{\lambda+1}$ does not decrease heights.

If $\lambda \in$ LORD, let $\psi^p_\lambda = \bigcup\limits_{\kappa<\lambda} \psi^p_\kappa$. Again we have $\psi^p_\lambda : \bigcup\limits_{\kappa<\lambda} M_\kappa = M_\lambda \to V$ such that $\psi^p_\lambda \alpha = \Phi_p \restriction M_\lambda$ and ψ^p_λ does not decrease heights.

Now let $\psi_p := \psi_\mu^p : A(p, \langle a \rangle) \to V$. Lemma 2.3.5 implies the existence of $\Psi : A \to V$ such that $\Psi\alpha = \Phi$. Hence A is balanced–projective. $\square$

We will need the following important lemma of Hunter to complete the characterization of balanced–projective groups.

Lemma 2.3.7. *For every height matrix M there exists a group $A \in \mathcal{A}$ such that there is $a \in A$ with $\mathrm{o}(a) = \infty$, $\mathbb{H}(a) = M$ and $A/\langle a \rangle \cong H$ for some totally projective group H.*

Proof. For the proof see [Hu, Proposition 5.20].

Theorem 2.3.8. *There are enough balanced–projectives. Every group G can be embedded into a balanced–exact sequence*

$$0 \to B \to A \to G \to 0$$

where $A \in \mathcal{A}^\Sigma$.

Proof.
For every element $g \in G$ we choose a group $A_g \in \mathcal{A}$ such that there is $a_g \in A_g$ with $\mathrm{o}(a_g) = \mathrm{o}(g)$ and $\mathbb{H}(a_g) = \mathbb{H}(g)$. If $\mathrm{o}(g) = \infty$, then Lemma 2.3.7 shows that such a group A_g exists. If $\mathrm{o}(g)$ is finite, choose a direct sum of generalized Prüfer groups such that the desired element exists.
If $\mathrm{o}(a_g) = \infty$, then by Lemma 2.3.1 $\langle a_g \rangle$ is p–nice in $A_g(p, \langle a_g \rangle)$. If $\mathrm{o}(a_g)$ is finite, $\langle a_g \rangle$ is p–nice in $A_g(p, \langle a_g \rangle)$ as a cyclic subgroup of a torsion group (cf. Lemma 1.3.12 (a)). Let $\Phi^* : \langle a_g \rangle \to \langle g \rangle$ be the height–preserving homomorphism that maps a_g onto g. We have $A_g(p, \langle a_g \rangle) \cong \mathrm{T_p}(A_g/\langle a_g \rangle)$. By Lemma 1.3.12 (c) Φ^* can be extended to a homomorphism $\Phi_p^* : A_g(p, \langle a_g \rangle) \to G$ for every prime p. Lemma 2.3.5 shows that there is a homomorphism $\Phi_g : A_g \to G$ such that $\Phi_g \restriction \langle a_g \rangle = \Phi^*$. Now define $\Phi := \bigoplus_{g \in G} \Phi_g : \bigoplus_{g \in G} A_g \to G$. It is $a_g \Phi = g$ and Φ satisfies condition (c) of Lemma 1.3.21. Hence Φ is balanced and $\bigoplus_{g \in G} A_g =: A \in \mathcal{A}^\Sigma$. $\square$

Corollary 2.3.9. *$\bar{A}$ is the class of all balanced–projective groups.*

Proof.
If P is balanced–projective, then by Theorem 2.3.8 there exists some $A \in \mathcal{A}^\Sigma$ such that

$$0 \to K \to A \to P \to 0$$

is balanced–exact. Hence $A \cong P \oplus K$ and therefore $P \in \bar{\mathcal{A}}$.
This shows together with Theorem 2.3.6 that $\bar{\mathcal{A}}$ is the set of all balanced–projective groups. □

The next proposition summarizes the characterizations for torsion and torsion–free balanced–projective groups.

Proposition 2.3.10.

(a) *A torsion group T is balanced–projective if and only if it is totally projective.*

(b) *A torsion–free group C is balanced–projective if and only if it is completely decomposable.*

(c) *If A is balanced–projective, then $A/\operatorname{T}(A)$ is completely decomposable.*

(d) *A torsion summand of a balanced–projective group is totally projective.*

Proof.
(a) Of course, every totally projective group is balanced–projective. Now assume that T is balanced–projective. If $T \in \mathcal{A}$, T is totally projective by Lemma 2.3.3. If $T \in \mathcal{A}^{\Sigma}$, then $T = \bigoplus T_i$ where each T_i is totally projective and hence T is totally projective. If $T \in \bar{\mathcal{A}}$, there is a (torsion) group K such that $T \oplus K \in \mathcal{A}^{\Sigma}$ and $T \oplus K$ is totally projective. Therefore, T is totally projective by Lemma 1.3.11.
(b) Obviously, every completely decomposable group is balanced–projective. Hence assume that C is balanced–projective. If $C \in \mathcal{A}$, C is a rank 1 group. If $C \in \mathcal{A}^{\Sigma}$, then $C = \bigoplus C_i$ where $C_i \in \mathcal{A}$. Hence C is a direct sum of rank 1 groups, i.e. completely decomposable. If $C \in \bar{\mathcal{A}}$, C is a direct summand of a completely decomposable group and by Lemma 1.1.5 itself completely decomposable.
(c) Let A be balanced–projective. Then there is a group K such that $A \oplus K = \bigoplus_{i\in I} A_i$ with $A_i \in \mathcal{A}$. Hence $\operatorname{T}(A) \oplus \operatorname{T}(K) = \operatorname{T}(\bigoplus_{i\in I} A_i) = \bigoplus_{i\in I} \operatorname{T}(A_i)$ and therefore, $A/\operatorname{T}(A) \oplus K/\operatorname{T}(K) \cong \bigoplus_{i\in I} A_i/\operatorname{T}(A_i)$ where $A_i/\operatorname{T}(A_i)$ is of rank 1 for every $i \in I$. Whence $A/\operatorname{T}(A)$ is a direct summand of a completely decomposable group and by Lemma 1.1.5 itself completely decomposable.
(d) If T is a torsion summand of a balanced–projective group, then T itself is balanced–projective and by (a) totally projective.

□

2.4 The existence of B–splitters which are not splitters

A group A is called *splitter* if $\mathrm{Ext}(A, A) = 0$. In analogy to this we have

Definition 2.4.1. A group A is called *B–splitter* if $\mathrm{Bext}(A, A) = 0$.

Of course, every splitter is also a B–splitter. Moreover, the balanced–injective and balanced–projective groups are the trivial B–splitters. In this section we will construct non–trivial B–splitters which are not splitters. Therefore, we transfer the results of Eklof and Trlifaj [ET] for Ext to Bext.
First we need a lemma which shows that certain restrictions of a balanced epimorphism are again balanced.

Lemma 2.4.2. *Let*

$$0 \to C \to N \xrightarrow{\pi} A \to 0$$

be a balanced–exact sequence and B a balanced subgroup of A. Let N_0 denote the inverse image of B in N under π. Then the sequence

$$0 \to C \to N_0 \xrightarrow{\pi \restriction N_0} B \to 0$$

is balanced–exact, too.

Proof.
Without loss of generality we can assume that A, B, C and N are reduced. The following diagram commutes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C \arrow[r] \arrow[d] & N_0 \arrow[r, "\pi\restriction N_0"] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[d] & N \arrow[r, "\pi"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & N/N_0 \arrow[r, "\tilde{\pi}"] \arrow[d] & A/B \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

and the third row is exact by the 3×3–lemma. Now let $x \in B$. We have to show that there is $n \in N_0$ such that $\mathbb{H}^{N_0}(n) = \mathbb{H}^B(x)$ and $n\pi = x$. First we show that N_0 is balanced in N. Let $y \in N/N_0$. Then $\mathbb{H}^{N/N_0}(y) = \mathbb{H}^{A/B}(\tilde{y})$ where $\tilde{y} = y\tilde{\pi}$. Since B is balanced in A, there is $z \in A$ such that $\mathbb{H}^A(z) = \mathbb{H}^{A/B}(\tilde{y})$ and $z + B = \tilde{y}$. Moreover, there is $k \in N$ such that $\mathbb{H}^N(k) = \mathbb{H}^A(z) = \mathbb{H}^{N/N_0}(y)$ and $k\pi + B = \tilde{y}$. Then $k + N_0 = y$ and N_0 is balanced in N.
We always have $\mathbb{H}^B(x) \leq \mathbb{H}^A(x)$. There is some $n \in N \cap N_0$ such that $\mathbb{H}^N(n) = \mathbb{H}^A(x)$ and $n\pi = x$. Since N_0 is balanced in N, it is $\mathbb{H}^{N_0}(n) = \mathbb{H}^N(n) = \mathbb{H}^A(x) = \mathbb{H}^B(x)$. $\square$

Now we can prove the following

Proposition 2.4.3. *Let $A = A_\mu$ be the union of a continuous ascending chain of subgroups, $A = \bigcup_{\alpha<\mu} A_\alpha$, such that A_α is balanced in $A_{\alpha+1}$ for all $\alpha < \mu$. If* $\mathrm{Bext}(A_0, C) = 0$ *and* $\mathrm{Bext}(A_{\alpha+1}/A_\alpha, C) = 0$ *for all $\alpha < \mu$, then* $\mathrm{Bext}(A, C) = 0$.

Proof.
Without loss of generality we can assume that $\mu \in \mathrm{LORD}$ and that A and C are reduced. We will show by induction that $\mathrm{Bext}(A_\beta, C) = 0$ for all $\beta \leq \mu$.
First let $\beta = \gamma + 1$. The short exact sequence

$$0 \to A_\gamma \to A_{\gamma+1} \to A_{\gamma+1}/A_\gamma \to 0$$

is balanced and hence the sequence

$$0 = \mathrm{Bext}(A_{\gamma+1}/A_\gamma, C) \to \mathrm{Bext}(A_{\gamma+1}, C) \to \mathrm{Bext}(A_\gamma, C) = 0$$

is exact. Therefore, $\mathrm{Bext}(A_{\gamma+1}, C) = 0$.
Now let $\beta \in \mathrm{LORD}$. We have to show that every balanced–exact sequence

$$0 \to C \xrightarrow{\iota} N \xrightarrow{\pi} A_\beta \to 0$$

splits. Per transfinite induction on $\alpha < \beta$ we will define a continuous, increasing sequence of homomorphisms $\rho_\alpha : A_\alpha \to N$ such that $\rho_\alpha \pi = \mathrm{id}_{A_\alpha}$. Let ρ_0 be a splitting of $\pi\restriction(\pi^{-1}[A_0])$ (the corresponding sequence is balanced–exact by Lemma 2.4.2).
Now assume that ρ_α is already defined for all $\alpha < \tau < \beta$. If $\tau \in \mathrm{LORD}$, let $\rho_\tau = \bigcup_{\alpha<\tau} \rho_\alpha$. Now let $\tau = \gamma + 1$ and $\sigma : A_\tau \to N$ be a splitting of $\pi\restriction(\pi^{-1}[A_\tau])$. This splitting exists since $\mathrm{Bext}(A_\tau, C) = 0$ and the corresponding sequence is balanced–exact by Lemma 2.4.2. Since ρ_γ and $\sigma\restriction A_\gamma$ are both splittings of $\pi\restriction(\pi^{-1}[A_\gamma])$, we have $\rho_\gamma - (\sigma\restriction A_\gamma) : A_\gamma \to C\iota = \mathrm{Ker}\,\pi$.

Hence there is a homomorphism $\theta : A_\gamma \to C$ such that $\theta\iota = \rho_\gamma - (\sigma\restriction A_\gamma)$. We will show that θ can be extended to a homomorphism $\theta' : A_\tau \to C$. The following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & A_\gamma \arrow[r] \arrow[d, "\theta"'] & A_\tau \arrow[r] \arrow[d, "\Phi"'] & A_\tau/A_\gamma \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & C \arrow[r, "\psi"] & X \arrow[r] & A_\tau/A_\gamma \arrow[r] & 0
\end{tikzcd}$$

Here the first row is balanced–exact and X in the second row is constructed as push–out. We will show that also the second row is balanced. Since C and A are reduced, it remains to show that C is H–nice in X. Let $x \in A_\tau/A_\gamma$. Then there exists $y \in A_\tau$ such that $\mathbb{H}^{A_\tau}(y) = \mathbb{H}^{A_\tau/A_\gamma}(x)$ and y is mapped onto x. Since the diagram commutes, we have

$$\mathbb{H}^X(y\Phi) \geq \mathbb{H}^{A_\tau}(y) = \mathbb{H}^{A_\tau/A_\gamma}(x) \geq \mathbb{H}^X(y\Phi).$$

Therefore the second row is also balanced–exact and splits since $\mathrm{Bext}(A_\tau/A_\gamma, C) = 0$. Let $\xi : X \to C$ be the splitting map such that $\psi\xi = \mathrm{id}_C$. Then $\Phi\xi$ is a homomorphism from A_τ to C and $(\Phi\restriction A_\gamma)\xi = \theta$. Let $\theta' = \Phi\xi$ and define $\rho_\tau = \sigma + \theta'\iota$. Then ρ_τ is a splitting of $\pi\restriction(\pi^{-1}[A_\tau])$ since

$$\rho_\tau\pi = (\sigma + \theta'\iota)\pi = \sigma\pi + \theta'\iota\pi = \sigma\pi = \mathrm{id}_{A_\tau}$$

and ρ_τ extends ρ_γ.
If ρ_α is defined for all $\alpha < \beta$, then let $\rho_\beta = \bigcup\limits_{\alpha<\beta} \rho_\alpha$. Then ρ_β is a splitting of π and hence $\mathrm{Bext}(A_\beta, C) = 0$. □

For every group B there is a group A, which is the union of a balanced chain of subgroups, such that $\mathrm{Bext}(B, A) = 0$.

Theorem 2.4.4. *Let κ and λ be cardinals such that $\kappa \geq \aleph_0$ and $\lambda^\kappa = \lambda$. Moreover, let I be a set of cardinality $\leq \kappa$ and $\{B_i : i \in I\}$ a family of reduced groups of cardinality $\leq \kappa$ and L a reduced group of cardinality $\leq \kappa$. Then there is a group A of cardinality λ such that A is the union of a continuous ascending chain of subgroups $A = \bigcup\limits_{\alpha<\lambda} A_\alpha$ and*

(a) *$A_0 = L$;*

(b) *A_α is balanced in $A_{\alpha+1}$ for all $\alpha < \lambda$;*

(c) *$A_{\alpha+1}/A_\alpha \cong B_i$ for some $i \in I$;*

(d) *$\mathrm{Bext}(B_i, A) = 0$ for all $i \in I$.*

Proof.
Let
$$0 \to K_i \to X_i \to B_i \to 0$$
be a balanced–projective resolution of B_i for $i \in I$. Then $|X_i| = |B_i| \leq \kappa$ and hence $|K_i| \leq \kappa$ for all $i \in I$. We enumerate all set mappings from all K_is to λ by $\{\varphi_\alpha : \alpha < \lambda\}$ such that every map appears λ–times. Write $\lambda = \bigcup_{\alpha<\lambda} A_\alpha$ as a union of a continuous increasing chain of sets such that $|A_0| = |L|$ and $|A_{\alpha+1} \backslash A_\alpha| = |\alpha| \cdot \kappa$ for all $\alpha < \lambda$.
Inductively, we define a group structure on A_α. Let $A_0 \cong L$. Now assume that the group structure on A_α is already defined for all $\alpha < \beta$. If β is a limit ordinal, let $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ with the induced group structure.
Now let $\beta = \alpha + 1$. Then φ_α is a map from K_i into λ for some $i \in I$. We define a homomorphism $\varphi'_\alpha : K_i \to A_\alpha$ by $\varphi'_\alpha = \varphi_\alpha$ if φ_α is a homomorphism from K_i to A_α and $\varphi'_\alpha = 0$ otherwise. We define the group structure on $A_{\alpha+1}$ by the following push–out diagram.

$$\begin{tikzcd} K_i \arrow[r, "\mathrm{id}_{K_i}"] \arrow[d, "\varphi'_\alpha"'] & X_i \arrow[d, "\psi_\alpha"] \\ A_\alpha \arrow[r, "\varepsilon"] & A_{\alpha+1} \end{tikzcd}$$

Here $A_{\alpha+1} \cong (A_\alpha \oplus X_i)/H$ with $H = \{(k_i\varphi'_\alpha, -k_i) | k_i \in K_i\}$ and $A_{\alpha+1}/A_\alpha \cong B_i \cong X_i/K_i$. It is easy to check that A_α is balanced in $A_{\alpha+1}$.
Let $A = \bigcup_{\alpha<\lambda} A_\alpha$. Since
$$0 \to K_i \to X_i \to B_i \to 0$$
is balanced–exact, the sequence
$$\begin{aligned} 0 \to \mathrm{Hom}(B_i, A) \to \mathrm{Hom}(X_i, A) \to \mathrm{Hom}(K_i, A) \\ \to \mathrm{Bext}(B_i, A) \to \mathrm{Bext}(X_i, A) = 0 \end{aligned}$$
is exact. If we can show that every homomorphism from K_i to A extends to a homomorphism from X_i to A, then $\mathrm{Bext}(B_i, A) = 0$. Therefore, let φ be a homomorphism from K_i to A. Since $|K_i| \leq \kappa < \mathrm{cf}(\lambda)$, there is some $\beta \in \lambda$ such that $K_i\varphi \subseteq A_\beta$. Choose $\alpha \geq \beta$ such that $\varphi = \varphi_\alpha = \varphi'_\alpha$. In the construction we have shown that there is a homomorphism $\psi_\alpha : X_i \to A_{\alpha+1} \subseteq A$ such that $\psi_\alpha \restriction K_i = \varphi$. Hence $\mathrm{Bext}(B_i, A) = 0$. $\square$

Corollary 2.4.5. *Let L and A be the groups from Theorem* 2.4.4. *Then* $\mathrm{Bext}(L, A) = 0$ *implies* $\mathrm{Bext}(A, A) = 0$.

Proof.
If we assume that $\mathrm{Bext}(L, A) = 0$, then by Proposition 2.4.3 $\mathrm{Bext}(A, A) = 0$. $\square$

We will use this additional property of A for the construction of B–splitters.

Construction 2.4.6.
1. Let $L = \{0\}$ and choose a reduced group B which is not cotorsion. Let $\kappa \geq \max\{|B|, \aleph_0\}$ and choose a cardinal λ such that $\lambda^\kappa = \lambda$. Apply Theorem 2.4.4 to this setting. Then $\mathrm{Bext}(A, A) = 0$. Moreover, we claim that $\mathbb{Q} \oplus A$ is a B–splitter but not a splitter. First we show that $\mathrm{Bext}(\mathbb{Q} \oplus A, \mathbb{Q} \oplus A) = 0$. This follows from

$$\mathrm{Bext}(\mathbb{Q} \oplus A, \mathbb{Q} \oplus A) = \mathrm{Bext}(\mathbb{Q}, \mathbb{Q}) \oplus \mathrm{Bext}(\mathbb{Q}, A) \oplus \mathrm{Bext}(A, \mathbb{Q}) \oplus \mathrm{Bext}(A, A)$$

and the fact that $\mathbb{Q}$ is balanced–projective and A is a B–splitter.
If we can show that A is not cotorsion, then $\mathrm{Ext}(\mathbb{Q}, A) \neq 0$ and hence A is not a splitter. $A_1 \subseteq A$ is reduced and not cotorsion since $A_1/A_0 = A_1 \cong B$. By transfinite induction it follows that A_α is reduced for all $\alpha \leq \lambda$ since A_α is balanced in A_β for all $\alpha < \beta \leq \lambda$ by Lemma 1.3.26. Hence A is reduced. Moreover, A_α/A_1 and especially A/A_1 is reduced for all $1 < \alpha \leq \lambda$. If A is cotorsion, then by Lemma 1.2.2(a) A_1 is also cotorsion in contradiction to our choice of B. Therefore, A is not cotorsion and $\mathrm{Ext}(\mathbb{Q}, A) \neq 0$.

2. Let p be a prime and $L = \mathbb{Z}/p\mathbb{Z}$. Choose a group B which is not p–divisible. Let $\kappa \geq \max\{|B|, \aleph_0\}$ and choose a cardinal λ such that $\lambda^\kappa = \lambda$. Now we can apply Theorem 2.4.4. The group A is a B–splitter since L is balanced–projective. It remains to show that A is no splitter. Therefore, it is enough to show that A is not p–divisible because $\mathrm{Ext}(\mathbb{Z}/p\mathbb{Z}, A) = A/pA$. Since $A_0 = \mathbb{Z}/p\mathbb{Z}$ and B are not p–divisible and the A_α form a balanced chain, it follows by transfinite induction that A is not p–divisible. Hence $\mathrm{Ext}(A, A) \neq 0$.
Alternatively, one could choose $L = \bigoplus\limits_{p \in \Pi} \mathbb{Z}/p\mathbb{Z}$, B reduced and show that A is not divisible.

For the construction of the next two splitters we need a result of Salce for Ext.

Lemma 2.4.7. *Let S be a rational group and $A_S = \{x \in A | \chi_A(x) \geq \chi_S(1)\}$ for a group A. Then* $\mathrm{Ext}(S, A) = 0$ *if and only if A/A_S is cotorsion.*

Proof. For the proof see [Sa, Theorem 3.5].

Construction 2.4.8.
1. Let R and R' be rational groups of incomparable idempotent type not equal to $\mathbb{Q}$. Let $L = R \oplus R'$ and choose for B an indecomposable homogeneous group of type $\mathbb{Z}$ (for example we will construct such a group in Proposition 3.2.3). Then let $\kappa \geq \max\{|B|, \aleph_0\}$ and choose a cardinal λ such that $\lambda^\kappa = \lambda$. Again we apply Theorem 2.4.4. The group A is a B–splitter since L is balanced–projective. Hence it remains to show that $\operatorname{Ext}(A, A) \neq 0$. Therefore, we show that $\operatorname{Ext}(L, A) \neq 0$. It is $\operatorname{Ext}(L, A) = \operatorname{Ext}(R, A) \oplus \operatorname{Ext}(R', A)$. By Lemma 2.4.7 $\operatorname{Ext}(R, A) = 0$ if and only if A/A_R is cotorsion. For contradiction assume that A/A_R is cotorsion. Since B is homogeneous of type $\mathbb{Z}$ and the chain of the A_αs is balanced, $A_R = R$. The group $A/L = A/(R \oplus R') = A/(A_R \oplus R')$ is reduced because B and L are reduced and the chain of the A_αs is balanced. If A/A_R is cotorsion, then by Lemma 1.2.2(a) $(R' \oplus A_R)/A_R = (R' \oplus R)/R \cong R'$ is cotorsion, a contradiction. Hence A/A_R is not cotorsion and therefore, $\operatorname{Ext}(A, A) \neq 0$.

2. Let $L = R$ be a rational group of type $\mathbf{t} < t(\mathbb{Q})$. Moreover, choose a B_1–group B such that $B(\mathbf{t}) = \{0\}$ and that is not cotorsion. Then let $\kappa \geq \max\{|B|, \aleph_0\}$ and choose a cardinal λ such that $\lambda^\kappa = \lambda$. We apply Theorem 2.4.4. The constructed group A is a B–splitter and again a B_1–group since $A_{\alpha+1}/A_\alpha \cong B$ and hence $\operatorname{Bext}(A_{\alpha+1}/A_\alpha, T) = 0$ for all torsion groups T. Then Proposition 2.4.3 implies that $\operatorname{Bext}(A, T) = 0$ for all torsion groups T. It remains to show that $\operatorname{Ext}(A, A) \neq 0$. Again we use Lemma 2.4.7 and show that $\operatorname{Ext}(R, A) \neq 0$. It is $A_R = R$. As before A/R is reduced and $B \cong A_1/A_0 = A_1/R \subseteq A/R$. If A/R is cotorsion, then by Lemma 1.2.2(a) $A_1/R \cong B$ is cotorsion. Since this would contradict our choice of B, A/R is not cotorsion and hence $\operatorname{Ext}(A, A) \neq 0$.

3 B–cotorsion pairs

Salce [Sa] has introduced in 1979 the so–called cotorsion pairs. He has shown that there is a bijection from the set of all cotorsion pairs between the classical cotorsion pair (the pair generated by $\mathbb{Q}$) and the maximal one (the pair generated by the class of all abelian groups) to the power set of the set of all primes Π. Moreover, Salce has proved some important results on the cotorsion pairs generated by rational groups. With the help of these results Göbel, Shelah and Wallutis [GSW] have shown that the sublattice of all cotorsion pairs singly generated by a rational group is isomorphic to the lattice of all types. Furthermore, they have shown that there is an embedding from any power set into the lattice of all cotorsion pairs.
In this chapter we will transfer the results on cotorsion pairs to B–cotorsion pairs. We will give the definition of B–cotorsion pairs analogous to the definition of cotorsion pairs given by Salce [Sa]. In Section 3.2 we will show that the B–cotorsion pairs singly cogenerated by rational groups are all incomparable. Moreover, in Section 3.3 we will prove in analogy to the work by Göbel, Shelah and Wallutis [GSW] that every power set can be embedded into the lattice of all singly cogenerated B–cotorsion pairs. Hence there are ascending, descending and anti–chains of arbitrary length in the lattice of all singly cogenerated B–cotorsion pairs.

3.1 Definition and introduction of B–cotorsion pairs

Definition 3.1.1. Let $\mathcal{G}$ and $\mathcal{H}$ be classes of abelian groups. We call the pair $(\mathcal{G}, \mathcal{H})$ a *B–cotorsion pair* if the following hold.

(a) $\mathrm{Bext}(G, H) = 0$ for all $G \in \mathcal{G}$ and $H \in \mathcal{H}$;

(b) if $\mathrm{Bext}(G, X) = 0$ for all $G \in \mathcal{G}$, then $X \in \mathcal{H}$;

(c) if $\mathrm{Bext}(Y, H) = 0$ for all $H \in \mathcal{H}$, then $Y \in \mathcal{G}$.

This means that the pair $(\mathcal{G}, \mathcal{H})$ is maximal with respect to $\mathrm{Bext}(G, H) = 0$ for all $G \in \mathcal{G}$ and $H \in \mathcal{H}$.

This definition is the analog of the definition of cotorsion pairs (also called cotorsion theories) as given by Salce in [Sa]. Salce has shown that the cotorsion pairs form a complete lattice with respect to inclusion of the second component.
We will define an ordering of the B–cotorsion pairs by inclusion of the first component. For the reasons and advantages of this ordering see [GT, Definition 2.2.1].

Definition 3.1.2. Let $(\mathcal{G}, \mathcal{H})$ and $(\mathcal{G}', \mathcal{H}')$ be two B–cotorsion pairs.
Then let

$$(\mathcal{G}, \mathcal{H}) \leq (\mathcal{G}', \mathcal{H}')$$

if $\mathcal{G} \subseteq \mathcal{G}'$ or equivalently $\mathcal{H} \supseteq \mathcal{H}'$.

Note that also the notions of generated and cogenerated exchange in comparison to Salce [Sa].

Definition 3.1.3. Let $\mathcal{G}$ be a class of abelian groups. Then define

$$\mathcal{G}^{\perp_B} = \{X \,|\, \mathrm{Bext}(G, X) = 0 \text{ for all } G \in \mathcal{G}\}$$

and

$$^{\perp_B}\mathcal{G} = \{Y \,|\, \mathrm{Bext}(Y, G) = 0 \text{ for all } G \in \mathcal{G}\}.$$

$(^{\perp_B}(\mathcal{G}^{\perp_B}), \mathcal{G}^{\perp_B})$ is called the B–cotorsion pair *generated* by $\mathcal{G}$ and $(^{\perp_B}\mathcal{G}, (^{\perp_B}\mathcal{G})^{\perp_B})$ is called the B–cotorsion pair *cogenerated* by $\mathcal{G}$.

If $\mathcal{G}$ is a singleton, i.e. $\mathcal{G} = \{G\}$, then $(^{\perp_B}(\mathcal{G}^{\perp_B}), \mathcal{G}^{\perp_B})$ (resp. $(^{\perp_B}\mathcal{G}, (^{\perp_B}\mathcal{G})^{\perp_B})$) is called singly (co–) generated by G.

The B–cotorsion pairs form a complete lattice with respect to the ordering defined in Definition 3.1.2. The supremum of a set $\{(\mathcal{G}_i, \mathcal{H}_i) | i \in I\}$ of B–cotorsion pairs is given by $\bigvee\limits_{i \in I} (\mathcal{G}_i, \mathcal{H}_i) = (^{\perp_B}(\bigcap\limits_{i \in I} \mathcal{H}_i), \bigcap\limits_{i \in I} \mathcal{H}_i)$ and the infimum by $\bigwedge\limits_{i \in I} (\mathcal{G}_i, \mathcal{H}_i) = (\bigcap\limits_{i \in I} \mathcal{G}_i, (\bigcap\limits_{i \in I} \mathcal{G}_i)^{\perp_B})$.
As we have seen in Theorem 2.2.4 the supremum of the lattice of B–cotorsion pairs is the maximal B–cotorsion pair $(\mathrm{Mod}-\mathbb{Z}, \mathcal{AC})$ where $\mathrm{Mod}-\mathbb{Z}$ is the class of all abelian groups and $\mathcal{AC}$ is the class of all algebraically compact groups. By Corollary 2.3.9 the infimum is the minimal B–cotorsion pair $(\bar{\mathcal{A}}, \mathrm{Mod}-\mathbb{Z})$ where $\bar{\mathcal{A}}$ is the class defined in Definition 2.3.2.

3.2 Rational B–cotorsion pairs

In this section we want to characterize the lattice of the B–cotorsion pairs singly cogenerated by rational groups. If R is a rational group, then Bican and Fuchs [BF] have called a torsion–free group A an *R–group* if $\mathrm{Bext}(A, R) = 0$, i.e. if A is in $^{\perp_B}R$.

In the following let R always denote a rational group not equal to $\mathbb{Q}$, $\mathbf{t}$ its type and $\chi_R(1) = (t_p)_{p \in \Pi}$. Moreover, let R_0 denote the largest subgroup of R of idempotent type $\mathbf{t}_0$. Note that R_0 is a subring of $\mathbb{Q}$, denoted the nucleus of R and that $R_0 \cong \mathrm{End}(R)$ canonically. Define $\Pi_R = \{p \in \Pi : t_p = \infty\}$ and $\Pi_0 = \Pi \backslash \Pi_R$.

Proposition 3.2.1. *Let τ be a type with $\tau \not\leq \mathbf{t}$. Moreover, let A be a torsion–free group such that $A = A(\tau)$. Then* $\operatorname{Bext}(A, R) = 0$.

Proof.
Let

$$0 \to R \to G \xrightarrow{\phi} A \to 0$$

be a balanced extension of R by A. For every $a \in A$ there is $\psi_a : \langle a \rangle_* \to G$ such that $\psi_a \phi = \mathrm{id}_{\langle a \rangle_*}$.

$$\begin{tikzcd}
 & & & \langle a\rangle_* \arrow[dl, "\psi_a"'] \arrow[d, "\mathrm{id}"] & \\
0 \arrow[r] & R \arrow[r] & G \arrow[r, "\phi"] & A \arrow[r] & 0
\end{tikzcd}$$

Then $t(a\psi_a) = t(a)$. Let $g_a := a\psi_a$ and define $\psi : A \to G$ by $a\psi = g_a$. It remains to show that ψ is well–defined. Therefore, let $x = \sum_{k=1}^{n} r_k a_k$ with $x, a_k \in A$ and $r_k \in \mathbb{Z}$. Then $x\psi = g_x$ and $(\sum_{k=1}^{n} r_k a_k)\psi = \sum_{k=1}^{n} r_k g_{a_k}$. The difference $g_x - \sum_{k=1}^{n} r_k g_{a_k}$ has to be an element of R and hence has the same type as R unless it is 0. On the other hand x and a_k have type $\geq \tau \not\leq \mathbf{t}$ and hence $t(g_x - \sum_{k=1}^{n} r_k g_{a_k}) \geq \tau \not\leq \mathbf{t}$. This is a contradiction unless $g_x - \sum_{k=1}^{n} r_k g_{a_k} = 0$ and therefore, ψ is well–defined. Hence $\operatorname{Bext}(A, R) = 0$. $\square$

Corollary 3.2.2. *Let τ be a type with $\tau \not\leq \mathbf{t}$. Moreover, let $A/\operatorname{T}(A) = (A/\operatorname{T}(A))(\tau)$. Then* $\operatorname{Bext}(A, R) = 0$.

Proof.
Let

$$0 \to R \to G \xrightarrow{\phi} A \to 0$$

be a balanced–exact sequence. Then the following diagram commutes.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathrm{T}(G) \arrow[r] \arrow[d] & \mathrm{T}(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_R"'] & G \arrow[r] \arrow[d, "\gamma"'] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r, "\beta"] \arrow[d] & G/\mathrm{T}(G) \arrow[r] \arrow[d] & A/\mathrm{T}(A) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The last row is balanced–exact and splits because of Proposition 3.2.1. Hence there is a homomorphism $\phi : G/\mathrm{T}(G) \to R$ such that $\beta\phi = \mathrm{id}_R$. Then $\gamma\phi$ is a homomorphism from G to R such that $\alpha\gamma\phi = \mathrm{id}_R$. Therefore the second row splits and $\mathrm{Bext}(A, R) = 0$. $\square$

In the following proposition we construct a group $A^{(p)}$ which will play an important role in the characterization of the lattice of all B–cotorsion pairs singly cogenerated by rational groups. This group appears in [F2, Chapter 88] where it is constructed as an example for a homogeneous, indecomposable group of rank ≥ 2.

Proposition 3.2.3. *Let p be a prime and $2 \leq k \in \mathbb{N}$. Then there exists a group $A^{(p)}$ with the following properties.*

(a) $\operatorname{rk} A^{(p)} = k$*;*

(b) $A^{(p)}$ *is homogeneous of type* $\mathbb{Z}$*;*

(c) $A^{(p)}$ *is indecomposable;*

(d) $A^{(p)} \otimes R$ *is indecomposable of type* $\mathbf{t}$ *if* $p \in \Pi_0$*;*

(e) $A^{(p)} \otimes R$ *is completely decomposable of type* $\mathbf{t}$ *if* $p \in \Pi_R$*.*

Proof.
Let $G = \mathbb{Q}^{(p)}a_1 \oplus \cdots \oplus \mathbb{Q}^{(p)}a_k$ be completely decomposable of type $(0, \ldots, 0, \infty, 0, \ldots)$, where ∞ is the p–th entry. Of course, G has rank k. We will define $A^{(p)}$ as a subgroup of

G. Let $\pi_1 = 1$ and choose $k-1$ algebraically independent p–adic units $\pi_2, \ldots, \pi_k$. Let $\pi_{i,n} = s_{i,0}+s_{i,1}p+\cdots+s_{i,n-1}p^{n-1}$ be the $(n-1)$–th partial sum of $\pi_i = s_{i,0}+s_{i,1}p+\cdots+s_{i,n}p^n+\ldots$, where $0 \leq s_{i,n} < p$. Define $x_n = p^{-n}(a_1 + \pi_{2,n}a_2 + \cdots + \pi_{k,n}a_k) \in G$ for all $n \in \mathbb{N}$. Let $A^{(p)} = \langle a_j, x_i : j \leq k, i \in \mathbb{N}\rangle \subseteq G$. Obviously, $\operatorname{rk} A^{(p)} = k$ and every element in $A^{(p)}$ is of the form

$$lx_n + l_2a_2 + \cdots + l_ka_k$$

for some $n \in \mathbb{N}$ and with $l, l_2, \ldots, l_k \in \mathbb{Z}$.
Next we will show that $\langle a_2, \ldots, a_k\rangle$ is pure in $A^{(p)}$. Assume that

$$p^m(lx_n + l_2a_2 + \cdots + l_ka_k) = m_2a_2 + \cdots + m_ka_k$$

for $l, l_2, \ldots, l_k, m_2, \ldots, m_k \in \mathbb{Z}$. Comparing coefficients shows that l has to be 0 and hence $\langle a_2, \ldots, a_k\rangle$ is pure in $A^{(p)}$.
Now we can show that $A^{(p)}$ is homogeneous of type $\mathbb{Z}$. Assume for contradiction that $p^{-n}(l_1a_1 + \cdots + l_ka_k) \in A^{(p)}$ for all $n \in \mathbb{N}$. Then $p^{-n}(l_1a_1 + \cdots + l_ka_k - l_1p^nx_n) \in A^{(p)}$. Substituting x_n gives

$$\begin{aligned} &p^{-n}(l_1a_1 + \cdots + l_ka_k - l_1a_1 - l_1\pi_{2,n}a_2 - \cdots - l_1\pi_{k,n}a_k) = \\ &\qquad p^{-n}((l_2 - l_1\pi_{2,n})a_2 + (l_3 - l_1\pi_{3,n})a_3 + \cdots + (l_k - l_1\pi_{k,n})a_k) \in A^{(p)}. \end{aligned}$$

Since $\langle a_2, \ldots, a_k\rangle$ is pure in $A^{(p)}$, we have $p^n|(l_i - l_1\pi_{i,n})$ for all $n \in \mathbb{N}$, i.e. the sequence $\{l_i - l_1\pi_{i,n}\}_{n\in\mathbb{N}}$ converges to zero. Hence $l_i = l_1\pi_i$ for $i = 2, \ldots, k$ and π_i is rational. This is a contradiction because we had chosen algebraically independent elements π_i. Therefore, the type of $A^{(p)}$ is $\mathbb{Z}$.
Next we want to show that $A^{(p)}$ is indecomposable. Therefore, we show that $\operatorname{End} A^{(p)} \subseteq \mathbb{Q}$. Then $A^{(p)}$ has to be indecomposable. Let $\eta \in \operatorname{End} A^{(p)}$ and $A_0 = \langle a_1, \ldots, a_k\rangle$. Without loss of generality we can assume that $A_0\eta \subseteq A_0$ (otherwise substitute η by $m\eta$ for some $m \neq 0$). The images of the $a_i (i \leq k)$ determine η uniquely.

$$\eta : a_i \mapsto \sum_{j=1}^{k} t_{ij}a_j, \quad t_{ij} \in \mathbb{Z}$$

Thus we have

$$x_n\eta = p^{-n}\sum_{i=1}^{k}\pi_{i,n}a_i\eta = p^{-n}\sum_{j=1}^{k}\sum_{i=1}^{k}\pi_{i,n}t_{ij}a_j = l_nx_n + m_{2,n}a_2 + \cdots + m_{k,n}a_k$$

for some $l_n, m_{2,n}, \ldots, m_{k,n} \in \mathbb{Z}$. Comparing coefficients shows that

$$\sum_{i=1}^{k}\pi_{i,n}t_{i1} = l_n \text{ and } p^{-n}\sum_{j=2}^{k}(\sum_{i=1}^{k}\pi_{i,n}t_{ij} - l_n\pi_{j,n})a_j \in \langle a_2, \ldots, a_k\rangle\,.$$

Hence $\sum_{i=1}^{k}\pi_{i,n}t_{ij} - l_n\pi_{j,n}$ is divisible by p^n for all n. Letting $n \to \infty$ we get

$$\sum_{i=1}^{k}\pi_i t_{ij} - (\sum_{i=1}^{k}\pi_i t_{i1})\pi_j = 0.$$

Since the $\pi_i s$ are algebraically independent, $t_{jj} = t_{11}$ and $t_{ij} = 0$ for $i \neq j$. Hence η is multiplication by t_{11} and $\operatorname{End} A^{(p)} \subseteq \mathbb{Q}$.

In the same way one can show that $A^{(p)} \otimes R$ is indecomposable if $p \in \Pi_0$.

Now assume that $p \in \Pi_R$. We will show that $A^{(p)} \otimes R$ is homogeneous completely decomposable of type $\mathbf{t}$. Therefore, we show that $A^{(p)} \otimes R = A_0 \otimes R$. Since the inclusion "$\supseteq$" is clear, we only need to show that $x_n \otimes r \in A_0 \otimes R$ for all $n \in \mathbb{N}$ and all $r \in R$. It is $x_n = p^{-n}(a_1 + \pi_{2,n}a_2 + \cdots + \pi_{k,n}a_k)$ and hence

$$x_n \otimes r = (a_1 + \pi_{2,n}a_2 + \cdots + \pi_{k,n}a_k) \otimes p^{-n}r \in A_0 \otimes R.$$

Since A_0 is free of rank k, we have $A^{(p)} \otimes R \cong \bigoplus_k R$. □

We will need the following results of Bican and Fuchs.

Lemma 3.2.4. *Let A be a torsion–free group which elements are all of types $\leq \mathbf{t}$. Then the following hold.*

(a) $\operatorname{Bext}(A, R) = 0$ *if and only if* $\operatorname{Bext}(A \otimes R_0, R) = 0$.

(b) *If $A \otimes R_0$ is a B_2–group, then* $\operatorname{Bext}(A \otimes R_0, R) = 0$.

(c) *If A is countable and* $\operatorname{Bext}(A, R) = 0$, *then $A \otimes R_0$ is a B_2–group.*

(d) *Let $V = L$ hold. Then* $\operatorname{Bext}(A, R) = 0$ *if and only if $A \otimes R_0$ is a B_2–group.*

Proof. See [BF, Lemma 1.6, Corollary 2.4, Theorem 4.5, Theorem 7.1].

Proposition 3.2.5. *Let $R \neq \mathbb{Q} \neq R'$ be rational groups with $t(R) \neq t(R')$ and $(\mathcal{A}, \mathcal{R})$ and $(\mathcal{A}', \mathcal{R}')$ the B–cotorsion pairs cogenerated by R resp. R'. Then $(\mathcal{A}, \mathcal{R})$ and $(\mathcal{A}', \mathcal{R}')$ are incomparable.*

Proof.

Let R_0' be the largest idempotent type less than or equal to R'. First assume that the types of R and R' are incomparable. Then choose a prime p with $pR \neq R$ and a prime

p' with $p'R' \neq R'$. By Proposition 3.2.3 there are groups $A^{(p)}$ and $A^{(p')}$ such that both are of finite rank, indecomposable and homogeneous of type $\mathbb{Z}$. Moreover, $A^{(p)} \otimes R$ is indecomposable, homogeneous of type R and $A^{(p')} \otimes R'$ is indecomposable, homogeneous of type R'. By Lemma 1.1.7 a homogeneous group of finite rank is a B_2–group if and only if it is completely decomposable. Therefore, $\mathrm{Bext}(A^{(p)} \otimes R, R) \neq 0$ and $\mathrm{Bext}(A^{(p')} \otimes R', R') \neq 0$ by Lemma 3.2.4(c). On the other hand by Proposition 3.2.1, $\mathrm{Bext}(A^{(p)} \otimes R, R') = 0$ and $\mathrm{Bext}(A^{(p')} \otimes R', R) = 0$. Hence $A^{(p)} \otimes R \in \mathcal{A}' \backslash \mathcal{A}$ and $A^{(p')} \otimes R' \in \mathcal{A} \backslash \mathcal{A}'$, i.e. $(\mathcal{A}, \mathcal{R})$ and $(\mathcal{A}', \mathcal{R}')$ are incomparable.

Now consider the case that $t(R') < t(R_0) \leq t(R)$ (the case $t(R) < t(R_0') \leq t(R')$ is analogous). First choose a prime p such that $pR \neq R$. Again by Proposition 3.2.3 there is a group $A^{(p)}$ of finite rank that is indecomposable and homogeneous of type $\mathbb{Z}$. Moreover, $A^{(p)} \otimes R_0$ is indecomposable, homogeneous of type $R_0 (> t(R'))$. Hence by Proposition 3.2.1 $\mathrm{Bext}(A^{(p)} \otimes R_0, R') = 0$. On the other hand $A^{(p)} \otimes R_0$ cannot be a B_2–group by Lemma 1.1.7 and by Lemma 3.2.4(c) $\mathrm{Bext}(A^{(p)} \otimes R_0, R) \neq 0$. Whence $A^{(p)} \otimes R_0 \in \mathcal{A}' \backslash \mathcal{A}$. For the other direction choose a prime p such that $pR' \neq R'$ and $pR = R$. Again construct $A^{(p)}$ as in Proposition 3.2.3. Then $A^{(p)} \otimes R_0'$ is indecomposable, homogeneous of type R_0' and hence not a B_2–group. By Lemma 3.2.4(c) and (a) $\mathrm{Bext}(A^{(p)}, R') \neq 0$. On the other hand $A^{(p)} \otimes R_0$ is completely decomposable of type R_0 and hence $\mathrm{Bext}(A^{(p)}, R) = 0$. Therefore, $A^{(p)} \in \mathcal{A} \backslash \mathcal{A}'$ and $(\mathcal{A}, \mathcal{R})$ and $(\mathcal{A}', \mathcal{R}')$ are incomparable.

For the last case assume that $t(R_0') = t(R_0) \leq t(R') < t(R)$ (the case $t(R_0) = t(R_0') \leq t(R) < t(R')$ is analogous). As in the second case we can show that there is a group $A^{(p)} \otimes R \in \mathcal{A}' \backslash \mathcal{A}$. For the other direction let $\widetilde{\Pi}$ be the set of all primes where the types of R and R' have different entries. Of course $\widetilde{\Pi}$ has to be infinite. Partition $\widetilde{\Pi}$ into three disjoint infinite subsets Π_1, Π_2, Π_3. Let R_i be the rational group where the 1 is as often divisible by p as in R' for $p \in \Pi \backslash \Pi_i$ and as in R for $p \in \Pi_i$. Then $t(R') < t(R_i) < t(R)$ and $t(R_1) \cap t(R_2) = t(R_1) \cap t(R_3) = t(R_2) \cap t(R_3) = t(R')$ and $\sup_{i \leq 3} t(R_i) = t(R)$. Define $\phi : R' \to \bigoplus_{i \leq 3} R_i$ by $1 \mapsto (1, 1, 1)$. Then

$$0 \to R' \xrightarrow{\phi} \bigoplus_{i \leq 3} R_i \to A \to 0$$

where $A = (\bigoplus_{i \leq 3} R_i)/R'$, is exact, but does not split. It is easy to check that this sequence is balanced. Hence $\mathrm{Bext}(A, R') \neq 0$. To show that $\mathrm{Bext}(A, R) = 0$ we will show that every homomorphism $\psi : R' \to R$ lifts to a homomorphism $\widetilde{\psi} : \bigoplus_{i \leq 3} R_i \to R$. Then $\mathrm{Bext}(A, R) = 0$

by the long exact sequence

$$\cdots \to \mathrm{Hom}(\bigoplus_{i\leq 3} R_i, R) \to \mathrm{Hom}(R', R) \to \mathrm{Bext}(A, R) \to 0.$$

Now let ψ be a homomorphism from R' to R and $1\psi = r$. Choose a representation $r = \frac{r_1}{q}$ with $r_1 \in \mathbb{Z}, q \in \mathbb{N}$ such that r_1 and q have no common divisor. Write $q = p_1p_2p_3q_0$ with each p_i a product of primes in Π_i and q_0 a product of primes in $\Pi\backslash\widetilde{\Pi}$. Then $\frac{1}{p_1p_2p_3} = \frac{f_1}{p_1} + \frac{f_2}{p_2} + \frac{f_3}{p_3}$ with $f_i \in \mathbb{Z}\backslash\{0\}$ and f_i and p_i have no common divisor. Hence $r = \frac{r_1}{q} = \frac{r_1}{q_0}(\frac{f_1}{p_1} + \frac{f_2}{p_2} + \frac{f_3}{p_3})$. Now define $\widetilde{\psi} : \bigoplus_{i\leq 3} R_i \to R$ by $(1,0,0) \mapsto \frac{r_1f_2}{q_0p_2}$, $(0,1,0) \mapsto \frac{r_1f_3}{q_0p_3}$, $(0,0,1) \mapsto \frac{r_1f_1}{q_0p_1}$. Then $(1,1,1) \mapsto \frac{r_1f_1}{q_0p_1} + \frac{r_1f_2}{q_0p_2} + \frac{r_1f_3}{q_0p_3} = r$ and hence ψ lifts to $\widetilde{\psi}$ and $\mathrm{Bext}(A, R) = 0$. Thus we have shown that $A \in \mathcal{A}\backslash\mathcal{A}'$ and therefore, $(\mathcal{A}, \mathcal{R})$ and $(\mathcal{A}', \mathcal{R}')$ are incomparable. □

3.3 The lattice of B–cotorsion pairs

In analogy to Göbel, Shelah and Wallutis in [GSW] we will show in this section that every power set can be embedded into the lattice of all singly generated B–cotorsion pairs.
Throughout this section let I be a fixed set and denote by $\mathcal{P} = \mathcal{P}(I)$ its power set. Let $\kappa \geq |I|$ be a cardinal such that $\kappa = \mu^+$ for a cardinal μ with $\mu^{\aleph_0} = \mu$ (for example choose $\mu = |I|^{\aleph_0}$).
For all subsets $X, Y \subseteq I$ we will construct groups G_Y and H^X such that $\mathrm{Bext}(G_Y, H^X) = 0$ if and only if $Y \subseteq X$. The groups G_Y will be the same groups that were constructed by Göbel, Shelah and Wallutis in [GSW] using the Strong Black Box.

Lemma 3.3.1. *Let E be a stationary subset of κ. Then there exists a non–free group $G(E)$ with the following properties.*

(a) $|G(E)| = \kappa$*;*

(b) $G(E)$ *is* $\aleph_1$*–free;*

(c) $G(E)$ *is ultra–cotorsion–free;*

(d) *there is a* κ*–filtration* $\{G_\alpha : \alpha < \kappa\}$ *such that* $G(E)/G_\alpha$ *is* $\aleph_1$*–free if and only if* $\alpha \notin E$*.*

Proof. For the construction of $G(E)$ and the proof of the properties see [GT, Section 11.1].

The set $\kappa^o = \{\alpha \in \kappa : \mathrm{cf}(\alpha) = \omega\}$ is stationary in κ. By Lemma 1.4.3 there is a partition of κ^o into $|I|$ disjoint stationary subsets, $\kappa^o = \bigcup_{i \in I} E_i$. For $i \in I$ we define $G_i = G(E_i)$ as in Lemma 3.3.1. For a subset $Y \subseteq I$ let $G_Y = \bigoplus_{i \in Y} G_i$.

Definition 3.3.2. Let E be a stationary subset of κ. We call a group A *locally E–free*, if, for any smooth ascending chain $\{K_\alpha : \alpha < \kappa\}$ of subgroups K_α of A with $|K_\alpha| < \kappa$ the set $\{\delta \in E : K_{\delta+1}/K_\delta$ is not $\aleph_1$–free$\}$ is not stationary in κ.

Lemma 3.3.3. *Let $i \neq j \in I$. Then $G_i = G(E_i)$ is locally E_j–free but not locally E_i–free.*

Proof. See [GT, Proposition 11.3.5].

Lemma 3.3.4. *Let A be a cotorsion–free locally E_i–free group for some $i \in I$. Then* $\mathrm{Hom}(G_i, A) = 0$.

Proof. See [GT, Corollary 11.3.6].

The next step is to construct groups H^X such that $\mathrm{Bext}(G_Y, H^X) = 0$ if $Y \subseteq X$.

Proposition 3.3.5. *For every non–empty subset X of I there is an $\aleph_1$–free group H^X of cardinality λ such that* $\mathrm{Bext}(G_Y, H^X) = 0$ *if $Y \subseteq X$. Moreover, there is a λ–filtration $\{H^X_\alpha : \alpha < \lambda\}$ of H^X such that H^X/H^X_α is $\aleph_1$–free for all $\alpha < \lambda$.*

Proof.
Let H^X be the group constructed in Theorem 2.4.4 with $\{G_i : i \in X\}$ as family of reduced groups of cardinality κ and $H^X_0 = L = \mathbb{Z}^{(\kappa)}$. Then $\mathrm{Bext}(G_i, H^X) = 0$ for all $i \in X$ and hence $\mathrm{Bext}(G_Y, H^X) = 0$ for all $Y \subseteq X$. Let $\{H^X_\alpha : \alpha < \lambda\}$ be the filtration from the theorem. It remains to show that H^X and H^X/H^X_α are $\aleph_1$–free. We will show by induction on $\beta < \lambda$ that H^X_β is $\aleph_1$–free. (The proof for H^X/H^X_α is the same.) H^X_0 is $\aleph_1$–free as a free group. If β is a limit ordinal and H^X_γ is $\aleph_1$–free for all $\gamma < \beta$, then H^X_β is $\aleph_1$–free as a union of $\aleph_1$–free groups. If $\beta = \gamma + 1$ for some γ and H^X_γ is $\aleph_1$–free, then $H^X_\beta/H^X_\gamma \cong G_i$ for some $i \in X$. Hence H^X_β is $\aleph_1$–free as an extension of an $\aleph_1$–free group by an $\aleph_1$–free group. Thus we have shown that the desired group H^X exists for every subset X of I. $\square$

Lemma 3.3.6. *Let X be a non–empty subset of I and $i \in I \setminus X$. Then H^X and H^X/H^X_0 are locally E_i–free.*

Proof. The proof is the same as in [GT, Proposition 11.3.8].

It remains to show that $\mathrm{Bext}(G_Y, H^X) \neq 0$ if $Y \nsubseteq X$.

Proposition 3.3.7. *Let $\emptyset \neq X \subseteq I$, $i \in I \backslash X$ and H^X be the group constructed above. Then* $\mathrm{Bext}(G_i, H^X) \neq 0$.

Proof.
Let

$$0 \to K_i \to X_i \to G_i \to 0$$

be a balanced–projective resolution of G_i with $|K_i| = |X_i| = \kappa$. Since G_i is $\aleph_1$–free, X_i and K_i are free groups. Then the sequence

$$0 \to \mathrm{Hom}(G_i, H^X) \to \mathrm{Hom}(X_i, H^X) \to \mathrm{Hom}(K_i, H^X) \to \mathrm{Bext}(G_i, H^X) \to 0$$

is exact. In order to show that $\mathrm{Bext}(G_i, H^X) \neq 0$, it is enough to show that there is a homomorphism $\varphi : K_i \to H^X$ which does not extend to a homomorphism $\widetilde{\varphi} : X_i \to H^X$. Let $\varphi : K_i \to H_0^X \subseteq H^X$ be an isomorphism between the two free groups K_i and H_0^X. Assume for contradiction that there is an extension $\widetilde{\varphi} : X_i \to H^X$ of φ. Let $\overline{\varphi} : X_i/K_i \cong G_i \to H^X/H_0^X$ be the homomorphism induced by $\widetilde{\varphi}$. By Proposition 3.3.5 H^X/H_0^X is $\aleph_1$–free and in particular cotorsion–free. By Lemma 3.3.6 H^X/H_0^X is locally E_i–free. Then $\mathrm{Hom}(G_i, H^X/H_0^X) = 0$ by Lemma 3.3.4 . Hence $\overline{\varphi} = 0$ and $X_i\widetilde{\varphi} = H_0^X$. Therefore, the sequence

$$0 \to \mathrm{Ker}\,\widetilde{\varphi} \to X_i \xrightarrow{\widetilde{\varphi}} H_0^X \to 0$$

splits and $X_i = K_i \oplus \mathrm{Ker}\,\widetilde{\varphi}$. Hence $\mathrm{Ker}\,\widetilde{\varphi} \cong X_i/K_i \cong G_i$ is free, contradicting the fact that G_i is not free. Thus there is no extension $\widetilde{\varphi}$ of φ and $\mathrm{Bext}(G_i, H^X) \neq 0$. □

Now we can prove the main result of this section.

Theorem 3.3.8. *There is an embedding from $(\mathcal{P}, \subseteq)$ into the lattice of all B–cotorsion pairs $(\mathcal{B}, \leq)$.*

Proof.
Define $\Phi : (\mathcal{P}, \subseteq) \to (\mathcal{B}, \leq)$ by $Y\Phi = ({}^{\perp_B}(G_Y^{\perp_B}), G_Y^{\perp_B}) \in \mathcal{B}$ for $Y \subseteq I$. Then for $Y' \subseteq Y$ we have $G_{Y'} \subseteq G_Y$ and hence

$${}^{\perp_B}(G_{Y'}^{\perp_B}) \leq {}^{\perp_B}(G_Y^{\perp_B}),$$

i.e. Φ is order preserving. To show that Φ is injective let $X \neq Y$ be subsets of I such that $X \backslash Y \neq \emptyset$. Then $H^Y \in G_Y^{\perp_B} \backslash G_X^{\perp_B}$ by Propositions 3.3.5 and 3.3.7. Hence

$$Y\Phi = ({}^{\perp_B}(G_Y^{\perp_B}), G_Y^{\perp_B}) \neq ({}^{\perp_B}(G_X^{\perp_B}), G_X^{\perp_B}) = X\Phi$$

if $X \neq Y$. $\square$

This Theorem shows that there are ascending, descending and anti–chains of arbitrary length in the lattice of B–cotorsion pairs.

4 R–Whitehead groups

In 1952 Whitehead asked the famous question if every group A which satisfies $\mathrm{Ext}(A,\mathbb{Z})=0$ is free. Therefore, a group G is called *Whitehead group* if $\mathrm{Ext}(G,\mathbb{Z})=0$ and, more general, an S–module M is called *Whitehead module* if $\mathrm{Ext}_S(M,S)=0$. Shelah [Sh1] has shown in 1974 that Whitehead's problem is undecidable in ZFC. He proved that assuming $V=L$ all Whitehead groups of cardinality $<\aleph_{\omega_1}$ are free. In contrast to that he constructed a non–free Whitehead group of cardinality $\aleph_1$ assuming Martin's axiom and $2^{\aleph_0}>\aleph_1$. One year later Shelah [Sh2] has shown with the help of the singular compactness theorem (cf. Lemma 1.4.6) that assuming $V=L$ every Whitehead group is free. 1980 Shelah [Sh3] translated Whitehead's problem into a combinatorial problem. He proved that there is a non–free Whitehead group of cardinality $\aleph_1$ if and only if there is a ladder system on a stationary subset of ω_1 which satisfies 2–uniformization.
In analogy to the definition of a Whitehead group we define a group G to be an *R–Whitehead group* if $\mathrm{Ext}(G,R)=0$. In Section 4.1 we will show that an R–Whitehead group is p–torsion–free for all $p\in\Pi_0$. As for Whitehead groups we will see that the characterization of R–Whitehead groups depends on the model of set theory. Assuming $V=L$ a torsion–free group G is an R–Whitehead group if and only if $G\subseteq\bigoplus R_0$. Moreover, there is a torsion–free R–Whitehead group G of cardinality $\aleph_1$ which is not a subgroup of $\bigoplus R_0$ if and only if there is a ladder system on a stationary set which satisfies 2–uniformization.

Let R be a rational group and R_0 its nucleus. Salce [Sa] has shown that for a mixed group G

$$\mathrm{Ext}(G,R)=0 \;\Leftrightarrow\; \mathrm{Ext}(\mathrm{T}(G),R)=0 \text{ and } \mathrm{Ext}(G/\mathrm{T}(G),R)=0.$$

Hence, in order to characterize R–Whitehead groups, we may consider torsion and torsion–free groups seperately.

4.1 The torsion case

In this section we will concentrate on torsion groups G.

Proposition 4.1.1. *Let G be a p–group. Then* $\mathrm{Ext}(G,R)=0$ *iff* $p\in\Pi_R$.

Proof.
The sequence

$$0\to R\to\mathbb{Q}\to\mathbb{Q}/R\to 0$$

is exact and $\mathbb{Q}/R \cong \bigoplus_{q \in \Pi_0} Z(q^\infty)$. Then

$$0 = \mathrm{Hom}(G, \mathbb{Q}) \to \mathrm{Hom}(G, \mathbb{Q}/R) \to \mathrm{Ext}(G, R) \to \mathrm{Ext}(G, \mathbb{Q}) = 0$$

is exact, too. Hence $\mathrm{Hom}(G, \mathbb{Q}/R) \cong \mathrm{Ext}(G, R)$.
First assume that $p \in \Pi_R$. Then $\mathrm{Hom}(G, \bigoplus_{q \in \Pi_0} Z(q^\infty)) = 0$ and therefore, $\mathrm{Ext}(G, R) = 0$.
Now assume that $\mathrm{Ext}(G, R) = 0$ and hence $\mathrm{Hom}(G, \bigoplus_{q \in \Pi_0} Z(q^\infty)) = 0$. If $p \notin \Pi_R$, then there is a non–zero homomorphism from G to $\bigoplus_{q \in \Pi_0} Z(q^\infty)$. Since this would contradict our assumption, p is an element of Π_R. □

The complete characterization of torsion groups G with $\mathrm{Ext}(G, R) = 0$ follows immediately.

Corollary 4.1.2. *Let G be a torsion group. Then* $\mathrm{Ext}(G, R) = 0$ *iff* $\mathrm{T_p}(G) = 0$ *for all* $p \notin \Pi_R$.

4.2 The countable case

From now on we will concentrate on torsion–free groups G. Obviously R_0 is a principal ideal domain and $G \otimes R_0$ is an R_0–module. In Lemma 1.1.3 we have shown that $\mathrm{Ext}_{\mathbb{Z}}(G, R) = 0$ exactly if $\mathrm{Ext}_{R_0}(G \otimes R_0, R) = 0$. In the following we will often switch between G as a group and $G \otimes R_0$ as R_0–module and skip the index $\mathbb{Z}$ resp. R_0 of Ext since it will be clear from the context.

Definition 4.2.1. We call a torsion–free group G *R_0–free*, if $G \subseteq \bigoplus R_0$, i.e. if $G \otimes R_0$ is a free R_0–module.

Proposition 4.2.2. *Let G be a rational group. Then* $\mathrm{Ext}(G, R) = 0$ *if and only if* $G \subseteq R_0$.

Proof.
Without loss of generality we may assume that $1 \in G \subseteq \mathbb{Q}$. We will use Lemma 2.4.7 with G instead of S.
First assume that $G \subseteq R_0$. We show that $\mathrm{Ext}(R_0, R) = 0$ since then G satisfies $\mathrm{Ext}(G, R) = 0$ as a subgroup of R_0. It is $R_{R_0} = R$ since $R_0 \subseteq R$. Hence $R/R_{R_0} = 0$ is cotorsion and therefore, $\mathrm{Ext}(R_0, R) = 0$.
For the other direction we will show that $\mathrm{Ext}(G, R) \neq 0$ whenever $G \not\subseteq R_0$. If $t(G)$ and $t(R)$ are incomparable or $t(R) < t(G)$, then $R_G = 0$ and therefore, $R/R_G = R$ is not cotorsion (remember $R \neq \mathbb{Q}$). Hence $\mathrm{Ext}(G, R) \neq 0$.

It remains to concentrate on the case where $t(R_0) < t(G) \leq t(R)$. Then there are infinitely many primes p such that $0 < g_p \neq \infty$ for $\chi_G(1) = (g_p)_{p \in \Pi}$. For these primes $\frac{1}{p^{-g_p+t_p+1}} \notin R_G$ holds. Hence $R/R_G \cong \bigoplus_p \mathbb{Z}(p^{g_p})$ is unbounded and therefore not cotorsion. Thus $\mathrm{Ext}(G, R) \neq 0$. □

For idempotent types the ordering of the rational groups is preserved when switching to the cotorsion pairs singly cogenerated by these rational groups.

Proposition 4.2.3. *Let S_0 be a rational group of idempotent type and $R \leq S_0$. Then $\mathrm{Ext}(G, R) = 0$ implies $\mathrm{Ext}(G, S_0) = 0$ for arbitrary groups G.*

Proof.
The sequence

$$0 \to R \to S_0 \to S_0/R \to 0$$

is exact and $S_0/R \cong \bigoplus_{p \in \Pi_S \setminus \Pi_R} Z(p^\infty)$. This implies the exactness of

$$\begin{aligned} 0 \to \mathrm{Hom}(G, R) \to \mathrm{Hom}(G, S_0) \to \mathrm{Hom}(G, S_0/R) \\ \to \mathrm{Ext}(G, R) \to \mathrm{Ext}(G, S_0) \to \mathrm{Ext}(G, S_0/R) = 0. \end{aligned}$$

Hence if $\mathrm{Ext}(G, R) = 0$, then $\mathrm{Ext}(G, S_0) = 0$, too. □

Theorem 4.2.4. *Let $\mathcal{R}_0$ be the set of all rational groups of idempotent type and $\mathcal{C}_{R_0}$ the set of all cotorsion pairs which are singly cogenerated by a rational group with idempotent type.*
Then there is an orderpreserving isomorphism $\Phi : (\mathcal{R}_0, \leq) \to (\mathcal{C}_{R_0}, \leq)$, given by $R_0\Phi \mapsto ({}^\perp R_0, ({}^\perp R_0)^\perp)$.

Proof.
Let $S_0, R_0 \in \mathcal{R}_0$. If $t(R_0) \leq t(S_0)$, then by Proposition 4.2.3 $\mathrm{Ext}(G, R_0) = 0$ implies $\mathrm{Ext}(G, S_0) = 0$ for any group G. Hence ${}^\perp R_0 \subseteq {}^\perp S_0$ and $({}^\perp R_0, ({}^\perp R_0)^\perp) \leq ({}^\perp S_0, ({}^\perp S_0)^\perp)$. If $t(R_0) < t(S_0)$, then $S_0 \in {}^\perp S_0 \setminus {}^\perp R_0$ by Proposition 4.2.2, i.e. the equality only holds if $R_0 = S_0$.
Now assume that $t(R_0)$ and $t(S_0)$ are incomparable. Then $R_0 \in {}^\perp R_0 \setminus {}^\perp S_0$ and $S_0 \in {}^\perp S_0 \setminus {}^\perp R_0$ by Proposition 4.2.2. Hence $({}^\perp R_0, ({}^\perp R_0)^\perp)$ and $({}^\perp S_0, ({}^\perp S_0)^\perp)$ are incomparable. □

This is an interesting result since we have seen in Proposition 3.2.5 that the B–cotorsion pairs which are singly cogenerated by rational groups are all incomparable.

We continue to characterize the cotorsion pairs which are singly cogenerated by a rational group without idempotent type. Therefore, we need a preparatory Lemma.

Lemma 4.2.5. *Let $A \subseteq B$ be torsion–free R_0–modules, A free and B/A bounded. Then B is free.*

Proof.
Since B/A is bounded, there exists an $n \in \mathbb{N}$ such that $nB \subseteq A \subseteq B$ and p does not divide n for all $p \in \Pi_R$. Since A is free, nB is free, too. Moreover, $nB \cong B$ and hence B is free. $\square$

Proposition 4.2.6. *Let G be a torsion–free group of finite rank. Then* $\mathrm{Ext}(G, R) = 0$ *if and only if G is R_0–free.*

Proof.
First assume that G is R_0–free. Then $G \otimes R_0$ is a free R_0–module and $\mathrm{Ext}_{R_0}(G \otimes R_0, R) = 0$. By Lemma 1.1.3 $\mathrm{Ext}(G, R) = 0$, too.
Now let $\mathrm{Ext}(G, R) = 0$ and $\mathrm{rk}(G) = n$. Assume that $G \otimes R_0$ is not free as R_0–module. Then choose a free R_0–module $F \subseteq G \otimes R_0$ of rank n. The sequence

$$0 \to F \to G \otimes R_0 \to (G \otimes R_0)/F \to 0$$

is exact and $T := (G \otimes R_0)/F$ is a torsion module. T has only non–trivial p–components for $p \in \Pi_0$ since for every $a \in G \otimes R_0$ with $pa \in F$ for some $p \in \Pi_R$, $\frac{1}{p}(pa) = a \in F$. If T is bounded, $G \otimes R_0$ is free by Lemma 4.2.5. Hence T has to be unbounded. We distinguish the following two cases.

(i) There is $p \in \Pi_0$ with $Z(p^\infty) \subseteq T$;

(ii) the socle $S(T)$ of T is infinite.

In the first case $J_p = \mathrm{Hom}(Z(p^\infty), Z(p^\infty)) \subseteq \mathrm{Hom}(T, \mathbb{Q}/R)$. In the second case $\prod\limits_{i<\omega} Z(p_i) = \mathrm{Hom}(\bigoplus\limits_{i<\omega} Z(p_i), \bigoplus\limits_{p\in\Pi_0} Z(p^\infty)) \subseteq \mathrm{Hom}(T, \mathbb{Q}/R)$ where the p_is are the orders of the elements in $S(T)$. Since $\mathrm{Ext}(T, R) \cong \mathrm{Hom}(T, \mathbb{Q}/R)$, we have in both cases $|\,\mathrm{Ext}(T, R)| \geq 2^{\aleph_0}$.
The sequence

$$\cdots \to \mathrm{Hom}(F, R) \to \mathrm{Ext}(T, R) \to \mathrm{Ext}(G \otimes R_0, R) = 0$$

is exact. But since F is finitely generated, $\mathrm{Hom}(F,R)$ is countable, a contradiction. Hence $G \otimes R_0$ has to be free, i.e. G is R_0–free. $\square$

With the help of Pontryagin's criterion we can generalize this result to groups of countable rank.

Corollary 4.2.7. *Let G be a torsion–free group of countable rank.
Then $\mathrm{Ext}(G,R)=0$ if and only if G is R_0–free.*

Proof.
If G is R_0–free, then $\mathrm{Ext}(G,R)=0$ as before. Now, let $\mathrm{Ext}(G,R)=0$. By Lemma 1.1.3 this is equivalent to $\mathrm{Ext}(G\otimes R_0,R)=0$. Then $\mathrm{Ext}(M,R)=0$ for every finite rank R_0–submodule M of $G\otimes R_0$. By Proposition 4.2.6 M is a free R_0–module. Since this holds for every finite rank R_0–submodule, $G\otimes R_0$ is $\aleph_1$–free by Lemma 1.4.5. Since $G\otimes R_0$ is countable, it is a free R_0–module and G is R_0–free. $\square$

4.3 R–Whitehead groups assuming V=L

In this section we will show that assuming $V=L$ a torsion–free group G is an R–Whitehead group if and only if it is R_0–free. L denotes Gödel's constructible universe. The assumption $V=L$ implies that the continuum hypothesis ($2^{\aleph_0}=\aleph_1$) holds. Moreover there are some prediction principles which hold assuming $V=L$. We will use the weak diamond which we define in Definition 4.3.4.
For the characterization of R–Whitehead groups we need the following result on Whitehead modules. Recall that a PID S is slender if every homomorphism $\varphi : \prod_{i<\omega} Se_i \to S$ maps almost all e_is $(i<\omega)$ to 0. For example, R_0 is a slender PID.

Lemma 4.3.1. *Let $V=L$ hold and S be a slender PID of cardinality $\leq \aleph_1$. Then every Whitehead module is free.*

Proof. For the proof see [EM, Corollary XII.1.11].

Corollary 4.3.2. *Let $V=L$ hold and G be a torsion–free group. Then $\mathrm{Ext}(G,R_0)=0$ if and only if G is R_0–free.*

Proof.
First assume that $G\subseteq \bigoplus R_0$. By Lemma 4.2.2 $\mathrm{Ext}(G,R_0)=0$ since the first component

is closed under direct sums and subgroups.
Now assume that $\mathrm{Ext}(G, R_0) = 0$. By Lemma 1.1.3 this is equivalent to $\mathrm{Ext}(G \otimes R_0, R_0) = 0$. Hence $G \otimes R_0$ has to be a Whitehead–module and by Lemma 4.3.1 $G \otimes R_0$ is free. $\square$

Corollary 4.3.3. *Let $V = L$ hold. If G is torsion–free and* $\mathrm{Ext}(G, R_0) = 0$*, then* $\mathrm{Ext}(G, T) = 0$ *for all $T \geq R_0$.*

Proof.
By Corollary 4.3.2 $\mathrm{Ext}(G, R_0) = 0$ implies that $G \subseteq \bigoplus R_0 \subseteq \bigoplus T_0$. Since the first component of Ext is closed under direct sums and subgroups, Lemma 4.2.2 shows that $\mathrm{Ext}(G, T) = 0$ for all $T \geq R_0$. $\square$

In order to generalize Corollary 4.3.2 to rational groups which are not of idempotent type, we will need the following prediction principle that is called the weak diamond. It was discovered by Devlin and Shelah [DS] and is a weakening of the $\diamondsuit$–principle which is due to Jensen (see [EM, Definition VI.1.1]).

Definition 4.3.4. Let κ be a regular uncountable cardinal and E a stationary subset of κ. By $\Phi_\kappa(E)$ we denote the following principle.
Let for every $\alpha \in E$ a function $P_\alpha : \mathcal{P}(\alpha) \to 2 = \{0, 1\}$ be given. Then there is a function $\rho : E \to 2$ such that $\{\alpha \in E : P_\alpha(X \cap \alpha) = \rho(\alpha)\}$ is stationary in κ for every $X \subseteq \kappa$.
$\Phi_\kappa(E)$ is called the *weak diamond*.

Remark 4.3.5. *$V = L$ implies $\Phi_\kappa(E)$ for every stationary subset of κ ($cf.$[EM]).*

We will not use the weak diamond itself but the following consequence of it.

Lemma 4.3.6. *Let $\Phi_\kappa(E)$ hold. Then for all sets A of cardinality κ, B of cardinality $\leq \kappa$, κ–filtrations $\{A_\nu : \nu \in \kappa\}$ and $\{B_\nu : \nu \in \kappa\}$ of A resp. B and every family of functions $P_\alpha : {}^{A_\alpha}B_\alpha \to 2$ $(\alpha \in E)$, there is a function $\rho : E \to 2$ such that for all functions $f : A \to B$ the set $\{\alpha \in E : P_\alpha(f \restriction A_\alpha) = \rho(\alpha)\}$ is stationary in κ.*

Proof. For the proof see [EM, Lemma VI.1.7].

We now prove the existence of a so–called associated free resolution of a module A and a given κ–filtration $\{A_\alpha : \alpha < \kappa\}$. Here we will only give the proof for PIDs. For arbitrary rings see [EM, Lemma XII.1.8].

Lemma 4.3.7. *Let κ be a regular uncountable cardinal and $\{A_\alpha : \alpha < \kappa\}$ a κ–filtration of the R_0–module A. Then there is a short exact sequence*

$$0 \to K \to F \xrightarrow{\varphi} A \to 0$$

where K and F are free R_0–modules, $F = \bigoplus_{\alpha<\kappa} F_\alpha, K = \bigoplus_{\alpha<\kappa} K_\alpha$ and $|F_\alpha|, |K_\alpha| < \kappa$ for all $\alpha < \kappa$. Moreover, there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{\beta<\alpha} K_\beta & \longrightarrow & \bigoplus_{\beta<\alpha} F_\beta & \xrightarrow{\varphi_\alpha} & A_\alpha & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & F & \xrightarrow{\varphi} & A & \longrightarrow & 0 \end{array}$$

with exact rows where the vertical arrows are inclusions.

Proof.
We will construct the modules F_α and K_α by induction on α. For $\alpha = 1$ choose a free resolution

$$0 \to K_0 \to F_0 \xrightarrow{\varphi_1} A_1 \to 0.$$

Assume that F_β, K_β and φ_γ are already defined for all $\beta < \gamma < \alpha$ such that φ_γ extends $\varphi_{\gamma'}$ for every $\gamma' < \gamma$.
If α is a limit ordinal, let $\varphi_\alpha = \bigcup_{\beta<\alpha} \varphi_\beta$. Then $\varphi_\alpha : \bigoplus_{\beta<\alpha} F_\beta \to A_\alpha$.
Now assume that $\alpha = \gamma + 1$. Choose a free module F_γ of cardinality $< \kappa$ such that there exists an epimorphism $\psi_\gamma : F_\gamma \to A_{\gamma+1}$. Define $\varphi_\alpha : \bigoplus_{\beta<\alpha} F_\beta \to A_{\gamma+1}$ by $\varphi_\alpha \restriction \bigoplus_{\beta<\gamma} F_\beta = \varphi_\gamma$ and $\varphi_\alpha \restriction F_\gamma = \psi_\gamma$. Choose a basis $\{b_i : i \in I\}$ of $\psi_\gamma^{-1}[A_\gamma]$ and $\{f_i : i \in I\} \subseteq \bigoplus_{\beta<\gamma} F_\beta$ such that $\varphi_\gamma(f_i) = \psi_\gamma(b_i)$. Let K_γ be the R_0–module generated by $\{b_i - f_i : i \in I\}$. Then

$$0 \to \bigoplus_{\beta<\alpha} K_\beta \to \bigoplus_{\beta<\alpha} F_\beta \xrightarrow{\varphi_\alpha} A_\alpha \to 0$$

is exact. Now, let $K = \bigoplus_{\alpha<\kappa} K_\alpha$, $F = \bigoplus_{\alpha<\kappa} F_\alpha$ and $\varphi = \bigcup_{\beta<\kappa} \varphi_\beta$. Obviously, K and F are free and the Lemma is shown. $\square$

In the next lemma we generalize a result from [EM]. We will use it to prove that assuming $V = L$ every R–Whitehead group is R_0–free.

Lemma 4.3.8. *Let F be a free R_0–module and H and K submodules of F such that F/H is free and* $\mathrm{Ext}(F/(H+K), R) \neq 0$.

Then there are homomorphisms $f_0, f_1 : K \to R$ such that there exists no homomorphism $h : H \to R$ which can simultaneously be extended to a homomorphism from F to R containing f_0 and to a homomorphism from F to R containing f_1.

Proof.
Let f_0 be the zero homomorphism. Since $F/(H+K) \cong (F/H)/((H+K)/H)$, the sequence

$$0 \to (H+K)/H \to F/H \to F/(H+K) \to 0$$

is exact. Then

$$\begin{aligned} 0 \to \mathrm{Hom}(F/(H+K), R) &\to \mathrm{Hom}(F/H, R) \\ &\to \mathrm{Hom}((H+K)/H, R) \to \mathrm{Ext}(F/(H+K), R) \to 0 \end{aligned}$$

is exact with $\mathrm{Ext}(F/(H+K), R) \neq 0$. Therefore, there is a homomorphism $g : (H+K)/H \to R$ which cannot be extended to a homomorphism from F/H to R. Now, let $\rho : K \to (H+K)/H$ be the canonical epimorphism and $f_1 = g \circ \rho : K \to R$.
Assume that $h_0, h_1 : F \to R$ are homomorphisms extending f_0 resp. f_1 and $h_0 \restriction H = h_1 \restriction H$. Then $\overline{h_1 - h_0} : F/H \to R$ is a well–defined homomorphism induced by $h_1 - h_0$. It is $(h_1 - h_0)\restriction H = 0$ and $(h_1 - h_0)\restriction K = g \circ \rho$. Hence $\overline{h_1 - h_0}$ extends g, a contradiction. Therefore, f_0 and f_1 are our desired homomorphisms. □

We are now in the position to prove our main result of this section.

Theorem 4.3.9. *Let $V = L$ hold and G be a torsion–free group. Then G is an R–Whitehead group if and only if G is R_0–free.*

Proof.
We have already shown that $\mathrm{Ext}(G, R) = 0$, if G is R_0–free. Now assume that not every torsion–free R–Whitehead group is R_0–free and let $A \in R_0 - Mod$ be a counter example of minimal cardinality κ. Then every submodule U of A of smaller cardinality satisfies $\mathrm{Ext}(U, R) = 0$ and hence is free. Therefore, A is κ–free. Corollary 4.2.7 implies that κ is uncountable and by Lemma 1.4.6 κ has to be regular.
Now choose a κ–filtration $\{A_\alpha : \alpha < \kappa\}$ of A and an associated free resolution $\{F_\alpha : \alpha < \kappa\}$ and $\{K_\alpha : \alpha < \kappa\}$ as in Lemma 4.3.7. Let $E = \{\alpha \in \kappa : A_{\alpha+1}/A_\alpha \text{ not free }\}$. Since A is not free, E is stationary in κ by Lemma 1.4.8. Because A was a counter example of

minimal cardinality, $\mathrm{Ext}(A_{\alpha+1}/A_\alpha, R) \neq 0$ for all $\alpha \in E$. Moreover,

$$A_{\alpha+1}/A_\alpha \cong \left(\bigoplus_{\beta<\alpha+1} F_\beta / \bigoplus_{\beta<\alpha+1} K_\beta \right) / \left(\bigoplus_{\beta<\alpha} F_\beta / \bigoplus_{\beta<\alpha} K_\beta \right)$$
$$\cong F_\alpha/K_\alpha \cong \left(\bigoplus_{\beta<\alpha+1} F_\beta \right) / \left(\bigoplus_{\beta<\alpha} F_\beta \oplus K_\alpha \right).$$

Next we apply Lemma 4.3.8 with $\bigoplus_{\beta<\alpha+1} F_\beta$ instead of F, $\bigoplus_{\beta<\alpha} F_\beta$ instead of H and K_α instead of K. Then there are homomorphisms $f_{0\alpha}, f_{1\alpha} : K_\alpha \to R$ with the properties stated in Lemma 4.3.8. Define for every $\alpha \in E$ a function $P_\alpha : {}^{\bigoplus_{\beta<\alpha} F_\beta} R \to 2$ in the following way. For a homomorphism $h : \bigoplus_{\beta<\alpha} F_\beta \to R$ let $P_\alpha(h) = 0$ if and only if h cannot be extended to a homomorphism from $\bigoplus_{\beta<\alpha+1} F_\beta$ to R that contains $f_{0\alpha}$. Hence if $P_\alpha(h) = 1$, then h cannot be extended to a homomorphism from $\bigoplus_{\beta<\alpha+1} F_\beta$ to R that contains $f_{1\alpha}$ (see Lemma 4.3.8). Since we assume $V = L$, the weak diamond $\Phi_\kappa(E)$ holds. Lemma 4.3.6 implies that there exists a function $\rho : E \to 2$ such that for every function $f : F \to R$ the set $\{\alpha \in E : P_\alpha(f{\restriction}\bigoplus_{\beta<\alpha} F_\beta) = \rho(\alpha)\}$ is stationary in κ. Now let $g = \bigoplus_{\alpha<\kappa} f_{\rho(\alpha)\alpha} : K \to R$. Suppose that g could be extended to a homomorphism $h : F \to R$. Choose α such that $P_\alpha(h{\restriction}\bigoplus_{\beta<\alpha} F_\beta) = \rho(\alpha)$. By definition of P_α the homomorphism $h{\restriction}\bigoplus_{\beta<\alpha} F_\beta$ cannot be extended to a homomorphism from $\bigoplus_{\beta<\alpha+1} F_\beta \to R$ that contains $f_{\rho(\alpha)\alpha}$. But h is such an extension, a contradiction. Hence g cannot be extended to a homomorphism from F to R, i.e. $\mathrm{Hom}(F, R) \to \mathrm{Hom}(K, R)$ is not an epimorphism and $\mathrm{Ext}(A, R) \neq 0$ which contradicts our assumption. Therefore, every R_0–module A satisfying $\mathrm{Ext}(A, R) = 0$ is free. Hence every torsion–free R–Whitehead group is R_0–free. □

4.4 Uniformization and the existence of non–$\mathbf{R_0}$–free R–Whitehead groups

In this section we will show that there is a torsion–free non–R_0–free R–Whitehead group of cardinality $\aleph_1$ if and only if there is a ladder system on a stationary subset of ω_1 which has the 2–uniformization property.

Let R be a rational group and $p \in \Pi_0$. Then let $R(p)$ be the rational group isomorphic to R such that p does not divide 1 in $R(p)$.

Definition 4.4.1. Let κ, σ be cardinals with $\sigma < \kappa$ and $E \subseteq \kappa$ such that for all $\delta \in E$ holds $\delta \in \mathrm{LORD}$ and $\mathrm{cf}(\delta) = \sigma$.

(a) Let $\delta \in E$. Then a function $\eta_\delta : \sigma \to \delta$ is called a *ladder on* δ if it is strictly increasing and has range cofinal in δ.

(b) A family $\eta = \{\eta_\delta : \delta \in E\}$ is called a *ladder system on* E if every η_δ is a ladder on δ.

(c) Let λ be a cardinal ≥ 2. A λ*–colouring* of a ladder system η on E is a family $c = \{c_\delta : \delta \in E\}$ such that each c_δ is a map from σ to λ.

(d) A *uniformization* of a colouring c of η is a pair (f, f^*) such that $f : \kappa \to \lambda$, $f^* : E \to \sigma$ and for every $\delta \in E$ and $\nu \geq f^*(\delta)$ the equality $f(\eta_\delta(\nu)) = c_\delta(\nu)$ holds.

(e) A ladder system η has the λ*–uniformization property* (λ–UP) if every λ–colouring of η can be uniformized.

Lemma 4.4.2. *If the ladder system η has the 2–uniformization property, then η also has the p–uniformization property for every finite p.*

Proof. See [EM, Lemma XIII.3.2].

Construction of non–$\mathbf{R_0}$–free R–Whitehead groups

Let $p \in \Pi_0$ and η be a ladder system on the stationary set $E \subseteq \lim(\omega_1)$. Let F_p be the free R_0–module with basis $\{x_i : i \in \omega_1\} \cup \{z_{\delta,k} : \delta \in E, k \in \omega\}$ and K_p the free submodule of F_p generated by $\{w_{\delta,k} : \delta \in E, k \in \omega\}$ with $w_{\delta,k} = z_{\delta,k} - pz_{\delta,k+1} + x_{\eta_\delta(k)}$. Define $G^p(\eta) = F_p/K_p$. We will show that $G^p(\eta)$ is a non–free Whitehead module and in particular an R–Whitehead group which is not R_0–free.

Proposition 4.4.3. *$G^p(\eta)$ is a non–free R_0–module, i.e. $G^p(\eta)$ is a non–R_0–free group.*

Proof.
By Lemma 1.4.8 we have to show that the Γ–invariant of $G^p(\eta)$ is not zero. Therefore, we choose a filtration $\{G_\alpha : \alpha < \omega_1\}$ where G_α is generated by $\{z_{\delta,k} + K_p : \delta \in E, \delta < \alpha,$ $k \in \omega\} \cup \{x_{\delta+1} + K_p : \delta + 1 < \alpha\}$. We will show that for all $\alpha \in E$ the module $G_{\alpha+1}/G_\alpha$ is not free. For ease of notation let M be freely generated by $\{z_k : k \in \omega\}$ and N by $\{z_k - pz_{k+1} : k \in \omega\}$. Then $G_{\alpha+1}/G_\alpha \cong M/N$ for all $\alpha \in E$.
We will use the fact that M/N is not free if and only if N is not a direct summand of M. Assume for contradiction that N is a direct summand of M. Then there is a homomorphism

$\varphi : M \to M$ with $\varphi(z_k - pz_{k+1}) = z_k$ for all $k \in \omega$. Let $\varphi(z_m) = \sum_{k\in\omega} r_{m,k} z_k$ with $r_{m,k} \in R_0$ and almost all $r_{m,k} = 0$. Then

$$\begin{aligned} z_m = \varphi(z_m - pz_{m+1}) &= \sum_{k\in\omega} (r_{m,k} - pr_{m+1,k}) z_k \\ &= \sum_{k\neq m} (r_{m,k} - pr_{m+1,k}) z_k + (r_{m,m} - pr_{m+1,m}) z_m. \end{aligned}$$

Comparing coefficients yields

$$r_{m,k} = pr_{m+1,k} \quad \text{if } k \neq m$$

and

$$r_{m,m} - pr_{m+1,m} = 1.$$

Hence

$$p^{m+1} r_{m+1,m} = p^m (r_{m,m} - 1) = r_{0,m} - p^m$$

for all m. Then there is $\hat{m} \in \mathbb{N}$ such that $r_{0,\hat{m}} = 0$ and

$$p^{\hat{m}+1} r_{\hat{m}+1,\hat{m}} = -p^{\hat{m}}.$$

Hence $-pr_{\hat{m}+1,\hat{m}} = 1$, a contradiction since 1 is not divisible by p in R_0. Therefore M/N is not free and $E \subseteq \Gamma(G^p(\eta))$. By Lemma 1.4.8 $G^p(\eta)$ is not free. $\square$

Lemma 4.4.4. *For every homomorphism $\psi : K_p \to R(p)$ there is a homomorphism $\theta : F_p \to R(p)$ such that for all $\delta \in E$ and $k \in \omega$*

$$(\theta - \psi)(w_{\delta,k}) \in \{0, \ldots, p-1\}.$$

Proof.
Let $\theta(x_i) = 0$ for all $i \in \omega_1$. For $\delta \in E$ we will define $\theta(z_{\delta,k})$ by induction on k. Let $\theta(z_{\delta,0}) = 0$ and assume that $\theta(z_{\delta,n})$ is already defined for all $n \leq k$. Then choose $\psi'(w_{\delta,k}) \in \{0, \ldots, p-1\}$ such that $\theta(z_{\delta,k}) - \psi(w_{\delta,k}) - \psi'(w_{\delta,k})$ is divisible by p in $R(p)$. Define

$$\theta(z_{\delta,k+1}) = \frac{\theta(z_{\delta,k}) - \psi(w_{\delta,k}) - \psi'(w_{\delta,k})}{p} \in R(p).$$

Then

$$\begin{aligned} \theta(w_{\delta,k}) &= \theta(z_{\delta,k}) - p\theta(z_{\delta,k+1}) + \theta(x_{\eta_\delta(k)}) \\ &= \theta(z_{\delta,k}) - \theta(z_{\delta,k}) + \psi(w_{\delta,k}) + \psi'(w_{\delta,k}) \\ &= \psi(w_{\delta,k}) + \psi'(w_{\delta,k}) \end{aligned}$$

and hence

$$(\theta - \psi)(w_{\delta,k}) = \psi'(w_{\delta,k}) \in \{0, \dots, p-1\}.$$

$\square$

Theorem 4.4.5. *If η has the 2–UP, then* $\mathrm{Ext}(G^p(\eta), R) = 0$.

Proof.
Without loss of generality we can assume that $1 \in R$ is not divisible by p (otherwise replace R by $R(p)$). By Lemma 4.4.2 η has also the p–uniformization property. To show that $\mathrm{Ext}(G^p(\eta), R) = 0$, we will show that every homomorphism from K_p to R extends to a homomorphism from F_p to R. Therefore, let a homomorphism $\psi : K_p \to R$ be given. By Lemma 4.4.4 there is $\theta : F_p \to R$ such that for all $\delta \in E, k \in \omega$

$$(\theta - \psi)(w_{\delta,k}) \in \{0, \dots, p-1\}.$$

Define a p–colouring $c_\delta(k) = (\theta - \psi)(w_{\delta,k})$. Since η has the p–UP, there are maps f, f^* such that $f(\eta_\delta(k)) = c_\delta(k) = (\theta - \psi)(w_{\delta,k})$ for all $k \geq f^*(\delta)$. Let $\varphi'(x_i) = f(i)$ for all $i \in \omega_1$ and $\varphi'(z_{\delta,k}) = 0$ for $k \geq f^*(\delta)$. Then

$$\varphi'(w_{\delta,k}) = \varphi'(z_{\delta,k}) - p\varphi'(z_{\delta,k+1}) + \varphi'(x_{\eta_\delta(k)}) = f(\eta_\delta(k)) = (\theta - \psi)(w_{\delta,k})$$

for $k \geq f^*(\delta)$. Let $\varphi(x_i) = (\theta - \varphi')(x_i)$ for $i \in \omega_1$ and $\varphi(z_{\delta,k}) = (\theta - \varphi')(z_{\delta,k})$ for $k \geq f^*(\delta)$. Then

$$\varphi(x_i) = -\varphi'(x_i) = -f(i)$$

and

$$\varphi(z_{\delta,k}) = \theta(z_{\delta,k}) - \varphi'(z_{\delta,k}) = \theta(z_{\delta,k})$$

if $k \geq f^*(\delta)$. Hence

$$\begin{aligned}
\varphi(w_{\delta,k}) &= \varphi(z_{\delta,k}) - p\varphi(z_{\delta,k+1}) + \varphi(x_{\eta_\delta(k)}) \\
&= \theta(z_{\delta,k}) - p\theta(z_{\delta,k+1}) - (\theta - \psi)(w_{\delta,k}) \\
&= \psi(w_{\delta,k})
\end{aligned}$$

for $k \geq f^*(\delta)$. For $k < f^*(\delta)$ we define $\varphi(z_{\delta,k})$ backwards by

$$\varphi(z_{\delta,k}) = \psi(w_{\delta,k}) + p\varphi(z_{\delta,k+1}) - \varphi(x_{\eta_\delta(k)}).$$

Then $\varphi(w_{\delta,k}) = \psi(w_{\delta,k})$ for all $\delta \in E, k \in \omega$ and φ is an extension of ψ to F_p. Hence $\mathrm{Ext}(G^p(\eta), R) = 0$. $\square$

Remark 4.4.6. *The special case $R = R_0$ in Theorem 4.4.5 shows that $G^p(\eta)$ is also a non–free Whitehead module.*

For every $p \in \Pi_0$ the group $G^p(\eta)$ is a non–R_0–free R–Whitehead group. If $p \neq q \in \Pi_0$, then $G^p(\eta)$ and $G^q(\eta)$ are non–isomorphic.

Lemma 4.4.7. *There is no isomorphism $\varphi : G^q(\eta) \to G^p(\eta)$ for $p \neq q \in \Pi_0$.*

Proof.
Assume for contradiction that there is an isomorphism $\varphi : G^q(\eta) \to G^p(\eta)$ for $p \neq q \in \Pi_0$. Choose ω_1–filtrations $\{G^p_\alpha : \alpha < \omega_1\}$ for $G^p(\eta)$ and $\{G^q_\alpha : \alpha < \omega_1\}$ for $G^q(\eta)$ as in the proof of Proposition 4.4.3. Then $G^p_{\alpha+1}/G^p_\alpha$ is p–divisible for all $\alpha \in E$ and $G^q_{\alpha+1}/G^q_\alpha$ is q–divisible for all $\alpha \in E$. Hence G^p_β/G^p_α contains a non–zero p–divisible subgroup for all $\beta > \alpha$ and G^q_β/G^q_α contains a non–zero q–divisible subgroup for all $\beta > \alpha$. Obviously G^p_β/G^p_α contains no non–zero q–divisible subgroup for all $\beta > \alpha$ and G^q_β/G^q_α contains no non–zero p–divisible subgroup for all $\beta > \alpha$.
The set $\{G^q_\alpha\varphi : \alpha < \omega_1\}$ is an ω_1–filtration of $G^p(\eta)$. Let $C = \{\alpha \in \omega_1 : G^p_\alpha = G^q_\alpha\varphi\}$. Then C is a cub since two ω_1–filtrations of $G^p(\eta)$ coincide on a cub. Let $\alpha < \beta \in E \cap C \neq \emptyset$. Then

$$G^p_\beta/G^p_\alpha = G^q_\beta\varphi/G^q_\alpha\varphi$$

and the group on the left hand side contains a non–zero p–divisible subgroup while the group on the right hand side contains no non–zero p–divisible subgroup. A contradiction. Hence there is no isomorphism $\varphi : G^q(\eta) \to G^p(\eta)$ for $p \neq q \in \Pi_0$. □

The existence of non–$\mathbf{R_0}$–free R–Whitehead groups implies 2–uniformization

Lemma 4.4.8. *Let H be an abelian group and Y, Y' finite subsets of H with $|Y|^2 < |Y'|$. Then there exists $b \in Y'$ such that Y and $b + Y$ are disjoint.*

Proof.
It is $|\{x - y : x, y \in Y\}| \leq |Y|^2 < |Y'|$. Hence there is $b \in Y' \setminus \{x - y : x, y \in Y\}$. Then $b + Y$ and Y are disjoint. □

Let $p \in \Pi_0$. For $x \in R(p)$ define $[x] := b$ with $b \in \{0, \dots, p-1\}$ and $b \equiv |x| \bmod pR(p)$. Moreover, let $j_p \in \mathbb{N}_0$ be maximal with p^{j_p} divides $1 \in R$. Then $[p^{j_p}m] = [p^{j_p}m']$ if and only if $|m| \equiv |m'| \bmod pR$.

Lemma 4.4.9. *Let $p \in \Pi_0$. Then there exist k_p^0 and $k_p^1 \in R$ and a function $\Phi_p : R/pR \to 2$ such that $\Phi_p(m + k_p^l + pR) = l$ $(l = 0, 1)$ for all $m \in R$ with $(2[p^{j_p}m] + 1)^2 < p$.*

Proof.
Let $k_p^0 = 0$, $H = Y' := R/pR$ and $Y = \{m + pR : (2[p^{j_p}m] + 1)^2 < p\}$. Then $|Y|^2 < |Y'|$ and we can apply Lemma 4.4.8. Hence there is $b \in Y'$ such that Y and $b + Y$ are disjoint. Choose $k_p^1 \in \{0, \frac{1}{p^{j_p}}, \dots, \frac{p-1}{p^{j_p}}\}$ with $k_p^1 + pR = b$. Then $k_p^0 + Y + pR$ and $k_p^1 + Y + pR$ are disjoint and we can define Φ_p with the desired property. □

Lemma 4.4.10. *Let $p \in \Pi_0$, $r \geq 0$ and $\mu = (\mu_1, \dots, \mu_r)$ be a sequence of elements in R_0. Then there exist $k_{p,\mu}^l \in R$ $(l = 0, 1)$ and a function $\Phi_{p,\mu} : R/pR \to 2$ with the property that if $m_0, \dots, m_r \in R$ with $(2[p^{j_p}m_i] + 1)^{2r+2} < p$ for all $i \leq r$, then $\Phi_{p,\mu}(m_0 + \sum\limits_{i=1}^{r} \mu_i m_i + k_{p,\mu}^l + pR) = l$ for $l = 0, 1$.*

Proof.
We apply Lemma 4.4.8 to $H = R/pR = Y'$ and

$$Y = \{m_0 + \sum_{i=1}^{r} \mu_i m_i + pR : (2[p^{j_p}m_i] + 1)^{2r+2} < p\}.$$

It is

$$|Y|^2 \leq (2 \max_{i \leq r} \left([p^{j_p}m_i]\right) + 1)^{2r+2} < p = |Y'|$$

and hence there exists $b \in Y'$ such that $b + Y$ and Y are disjoint. Let $k_{p,\mu}^0 = 0$ and choose $k_{p,\mu}^1 \in \{0, \dots, \frac{p-1}{p^{j_p}}\}$ with $k_{p,\mu}^1 + pR = b$. Then $k_{p,\mu}^0 + Y + pR$ and $k_{p,\mu}^1 + Y + pR$ are disjoint and we can define $\Phi_{p,\mu}$. □

Before we can proceed we need to define a strictly increasing sequence of integers in the following way. Let $p \in \Pi_0$ and a positive integer r be given. Let $t_0 = 0$. If t_{i-1} is already defined, let $t_i = t_{i-1} + d_i$ where d_i is the smallest integer such that

$$(2p^{t_{i-1}} + 1)^{2r+2} p^{2t_{i-1}} < p_i^{d_i}.$$

Moreover, for $x \in R(p)$ define $[x]_p^i := b$ where $b \in \{0, \dots, p^{t_i} - 1\}$ such that $b \equiv |x| \bmod p^{t_i}R(p)$.

Lemma 4.4.11. *Let $r \geq 0$ and $p \in \Pi_0$. Moreover let t_i $(i \in \mathbb{N}_0)$ be the sequence defined above. Then for every sequence of functions $\mu = (\mu_1, \dots, \mu_r)$ with $\mu_j : \omega \to R_0$ and every*

$i \geq 1$ there exist a function $\Phi_{i,\mu} : R(p)/pR(p) \to 2$ and integers $k^l_{n,\mu} \in \{0, \dots, p-1\}$ $(t_{i-1} \leq n < t_i, l = 0, 1)$ such that for all $m_0, \dots, m_r \in R(p)$ with $[m_j]^i_p \leq p^{t_{i-1}}$ and $k_\nu \in \{0, \dots, p-1\}$ for $\nu < t_{i-1}$

$$\Phi_{i,\mu}(m_0 + \sum_{j=1}^{r} \left(\sum_{\nu < t_i} p^\nu \mu_j(\nu) \right) m_j + \sum_{\nu < t_{i-1}} p^\nu k_\nu + \sum_{n=t_{i-1}}^{t_i - 1} p^n k^l_{n,\mu} + p^{t_i} R(p)) = l.$$

Proof.
Again we apply Lemma 4.4.8 with $H = R(p)/p^{t_i}R(p)$,

$$\begin{aligned} Y =& \{m_0 + \sum_{j=1}^{r} \left(\sum_{\nu < t_i} p^\nu \mu_j(\nu) \right) m_j + \sum_{\nu < t_{i-1}} p^\nu k_\nu + p^{t_i} R(p) : [m_j]^i_p \leq p^{t_{i-1}} \\ & \text{for all } j \leq r, k_\nu \in \{0, \dots, p-1\}\} \end{aligned}$$

and

$$Y' = \{ \sum_{n=t_{i-1}}^{t_i - 1} p^n x_n + p^{t_i} R(p) : x_n \in \{0, \dots, p-1\}\}.$$

Then $|Y| \leq (2p^{t_{i-1}} + 1)^{r+1} p^{t_{i-1}}$ and $|Y'| = p^{t_i - 1 - t_{i-1} + 1} = p^{d_i}$. Hence $|Y|^2 < |Y'|$ by definition of the t_is. By Lemma 4.4.8 there is $b \in Y'$ such that Y and $b + Y$ are disjoint. Then $b = \sum_{n=t_{i-1}}^{t_i - 1} p^n b_n + p^{t_i} R(p)$ where $b_n \in \{0, \dots, p-1\}$. Let $k^0_{n,\mu} = 0$ and $k^1_{n,\mu} = b_n$ $(t_{i-1} \leq n < t_i)$. Then we can define $\Phi_{i,\mu}$ with the above property. □

Lemma 4.4.12. *Let $\lambda = 2$ or $\lambda = \omega$ and $S \subseteq \omega_1$ a stationary set. Moreover, assume that there is a ω_1–filtration $\{B_\nu : \nu \in \omega_1\}$ of a set B of cardinality $\aleph_1$ and a family of functions $\eta_\gamma : \omega \to B_\gamma$ $(\gamma \in S)$ such that for every λ–colouring $\{c_\gamma : \gamma \in S\}$ there exists a pair (f, f^*) with $f : B \to \lambda$, $f^* : S \to \omega$ and for every $\gamma \in S$ and $n \geq f^*(\gamma)$ holds $f(\eta_\gamma(n)) = c_\gamma(n)$.*
Then there is a stationary subset S' of S with $\widetilde{S} = \widetilde{S'}$ and a ladder system on S' which has the λ–uniformization property.

Proof. For the proof see [EM, Lemma XIII.2.4].

Now, we can prove the main result of this section.

Theorem 4.4.13. *If A is a torsion–free, non–R_0–free R–Whitehead group of cardinality $\aleph_1$, then there exists a ladder system on a stationary set which has the 2–uniformization property.*

Proof.

Let A be a torsion–free, non–R_0–free R–Whitehead group of cardinality $\aleph_1$. Then $A \otimes R_0 =: \hat{A}$ is a non–free R_0–module of cardinality $\aleph_1$ with $\mathrm{Ext}(\hat{A}, R) = 0$. $\hat{A}$ is $\aleph_1$–free by Corollary 4.2.7 and the set $S' = \{\nu < \omega_1 : \hat{A}/A_\nu \text{ not } \aleph_1\text{–free } \}$ is stationary in ω_1 for every ω_1–filtration $\{A_\nu : \nu \in \omega_1\}$ of $\hat{A}$ by Lemma 1.4.8. Choose an ω_1–filtration $\{A_\alpha : \alpha \in \omega_1\}$ such that $A_{\alpha+1}/A_\alpha$ is not free whenever $\hat{A}/A_\alpha$ is not $\aleph_1$–free. By Pontryagin's criterion (see Lemma 1.4.5) we can assume that $A_{\gamma+1}/A_\gamma$ has finite rank for $\gamma \in S' = \{\alpha < \omega_1 : A_{\alpha+1}/A_\alpha \text{ not free } \}$. Moreover, by Lemma 1.4.2 we can assume that there is $r \in \omega$ such that

$$S = \{\gamma \in \omega_1 : A_{\gamma+1}/A_\gamma \text{ is not free of rank } r+1 \text{ and every subgroup of rank } r \text{ is free } \}$$

is stationary in ω_1.

First we will prove the result for a special case. Assume that S is as above with $r = 0$. Moreover, assume that for every $\gamma \in S$ the module $A_{\gamma+1}$ is generated over A_γ by the set $\{z_{\gamma,n} : n \in \omega\}$ and the relations

$$p_{\gamma,n} z_{\gamma,n+1} = z_{\gamma,0} + a_{\gamma,n}$$

where the $p_{\gamma,n} \in \Pi_0$ are all distinct and $a_{\gamma,n} \in A_\gamma$. (Note that this special case can only occur if $|\Pi_0| = \infty$.)

We define inductively a free resolution

$$0 \to K \to F \xrightarrow{\varphi} \hat{A} \to 0$$

of $\hat{A}$ as in Lemma 4.3.7. Assume that F_β, K_β and φ_α are already defined for $\beta < \alpha \le \gamma$ such that

$$0 \to \bigoplus_{\beta<\alpha} K_\beta \to \bigoplus_{\beta<\alpha} F_\beta \xrightarrow{\varphi_\alpha} A_\alpha \to 0$$

and φ_α extends $\varphi_{\alpha'}$ for all $\alpha' < \alpha$. If $\gamma \in S$, let $F_\gamma = \bigoplus_{n\in\omega} z_{\gamma,n} R_0$ and define $\psi_\gamma : F_\gamma \to A_{\gamma+1}$ by $z_{\gamma,n} \mapsto z_{\gamma,n}$. Let $\varphi_{\gamma+1} : \bigoplus_{\beta<\gamma+1} F_\beta \to A_{\gamma+1}$ be given by $\varphi_{\gamma+1} \restriction \bigoplus_{\beta<\gamma} F_\beta = \varphi_\gamma$ and $\varphi_{\gamma+1} \restriction F_\gamma = \psi_\gamma$. Then K_γ has a basis $\{w_{\gamma,n} : n \in \omega\}$ with

$$w_{\gamma,n} = z_{\gamma,0} - p_{\gamma,n} z_{\gamma,n+1} + a'_{\gamma,n}$$

where $\varphi_\gamma(a'_{\gamma,n}) = a_{\gamma,n}$. If $\gamma \notin S$, define F_γ, K_γ and $\varphi_{\gamma+1}$ as in Lemma 4.3.7. Finally let $K = \bigoplus_{\beta<\omega_1} K_\beta$, $F = \bigoplus_{\beta<\omega_1} F_\beta$ and $\varphi = \bigcup_{\beta<\omega_1} \varphi_\beta$.

Now let $B = \mathbb{Z} \times F$ and $B_\nu = \mathbb{Z} \times (\bigoplus_{\beta<\nu} F_\beta)$. For $\gamma \in S$ define $\eta_\gamma(n) = (p_{\gamma,n}, a'_{\gamma,n}) \in B_\gamma$. By

Lemma 4.4.9 there exist $k^0_{p_{\gamma,n}}$ and $k^1_{p_{\gamma,n}} \in R$ and a function $\Phi_{p_{\gamma,n}} : R/R(p) \to 2$ with the properties stated there. Let a 2–coloring $c = \{c_\gamma : \gamma \in S\}$ be given and define $\theta : K \to R$ by $\theta(w_{\gamma,n}) = k^{c_\gamma(n)}_{p_{\gamma,n}}$. Since $\mathrm{Ext}(\hat{A}, R) = 0$, there is an extension $\bar{\theta} : F \to R$. We define the uniformizing map f on B as follows

$$f((k,y)) = \begin{cases} \Phi_k(\bar{\theta}(y) + kR) & \text{if } k \in \Pi_0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we define

$$f^*(\gamma) = \min\{n \in \mathbb{N} | \forall m \geq n : p_{\gamma,m} > (2[\bar{\theta}(z_{\gamma,0})] + 1)^2\}.$$

Then for all $n \geq f^*(\gamma)$ holds $f(\eta_\gamma(n)) = c_\gamma(n)$:
It is

$$\bar{\theta}(a'_{\gamma,n}) = \theta(w_{\gamma,n}) - \bar{\theta}(z_{\gamma,0}) + p_{\gamma,n}\bar{\theta}(z_{\gamma,n+1})$$

and hence

$$\bar{\theta}(a'_{\gamma,n}) + p_{\gamma,n}R = k^{c_\gamma(n)}_{p_{\gamma,n}} - \bar{\theta}(z_{\gamma,0}) + p_{\gamma,n}R.$$

Therefore,

$$\begin{aligned} f(\eta_\gamma(n)) &= f((p_{\gamma,n}, a'_{\gamma,n})) \\ &= \Phi_{p_{\gamma,n}}(\bar{\theta}(a'_{\gamma,n}) + p_{\gamma,n}R) \\ &= \Phi_{p_{\gamma,n}}(k^{c_\gamma(n)}_{p_{\gamma,n}} - \bar{\theta}(z_{\gamma,0}) + p_{\gamma,n}R) \\ &= c_\gamma(n). \end{aligned}$$

Hence by Lemma 4.4.12 there is a ladder system with the 2–uniformization property. This completes the proof in our special case.

In the general case using Lemma 1.4.2 again and if necessary replacing $A_{\gamma+1}/A_\gamma$ by a submodule we can assume that one of the following two cases holds.

1) For all $\gamma \in S$ holds that $A_{\gamma+1}/A_\gamma$ has a free submodule L_γ/A_γ of rank r such that $A_{\gamma+1}/L_\gamma$ as abelian group has a type which p–entries are all 0 or 1 if $p \in \Pi_0$ and ∞ if $p \in \Pi_R$.

2) There is a prime $p \in \Pi_0$ such that for all $\gamma \in S$, $A_{\gamma+1}/A_\gamma$ has a free submodule L_γ/A_γ of rank r such that the entries of the type of $A_{\gamma+1}/A_\gamma$ as abelian group are ∞ for p and all $q \in \Pi_R$ and 0 otherwise.

Then $A_{\gamma+1}$ is generated over A_γ by a set $\{y_{\gamma,j} : j = 1,\dots,r\} \cup \{z_{\gamma,n} : n \in \omega\}$ modulo the following relations.

1) $p_{\gamma,n} z_{\gamma,n+1} = z_{\gamma,0} - \sum_{j=1}^{r} \mu_{\gamma,j}(n) y_{\gamma,j} + a_{\gamma,n} \quad (n \in \omega)$
   where $p_{\gamma,n}$ are different primes in Π_0 and $\mu_{\gamma,j}(n) \in R_0$, $a_{\gamma,n} \in A_\gamma$;

2) $p z_{\gamma,n+1} = z_{\gamma,n} - \sum_{j=1}^{r} \mu_{\gamma,j}(n) y_{\gamma,j} + a_{\gamma,n} \quad (n \in \omega)$
   where $\mu_{\gamma,j}(n) \in R_0$ and $a_{\gamma,n} \in A_\gamma$.

Again we construct a free resolution of $\hat{A}$ as in Lemma 4.3.7.

$$0 \to K \to F \xrightarrow{\varphi} \hat{A} \to 0$$

F_γ has $\{z_{\gamma,n} : n \in \omega\}$ as basis and K_γ has a basis $\{w_{\gamma,n} : n \in \omega\}$ where the $w_{\gamma,n}$ depend on the case.

1) $w_{\gamma,n} = z_{\gamma,0} - p_{\gamma,n} z_{\gamma,n+1} - \sum_{j=1}^{r} \mu_{\gamma,j}(n) y_{\gamma,j} + a'_{\gamma,n} \quad$ with $\varphi_\gamma(a'_{\gamma,n}) = a_{\gamma,n}$.

2) $w_{\gamma,n} = z_{\gamma,n} - p z_{\gamma,n+1} - \sum_{j=1}^{r} \mu_{\gamma,j}(n) y_{\gamma,j} + a'_{\gamma,n} \quad$ with $\varphi_\gamma(a'_{\gamma,n}) = a_{\gamma,n}$.

For the first case let $B = R_0^r \times \mathbb{Z} \times F$ and $B_\nu = R_0^r \times \mathbb{Z} \times (\bigoplus_{\beta<\nu} F_\beta)$. Define $\mu_\gamma(n) = \langle \mu_{\gamma,j}(n) : j = 1,\dots,r \rangle$ and $\eta_\gamma : \omega \to B$ by

$$\eta_\gamma(n) = \left\langle \mu_\gamma(n), p_{\gamma,n}, a'_{\gamma,n} \right\rangle .$$

By Lemma 4.4.10 there are $k^0_{p_{\gamma,n},\mu_\gamma(n)}$ and $k^1_{p_{\gamma,n},\mu_\gamma(n)} \in R$ and a function $\Phi_{p_{\gamma,n},\mu_\gamma(n)} : R/pR \to 2$ with the properties stated there. Let a 2–colouring $c = \{c_\gamma : \gamma \in S\}$ be given. Then define $\theta : K \to R$ by $\theta(w_{\gamma,n}) = k^{c_\gamma(n)}_{p_{\gamma,n},\mu_\gamma(n)}$. Since $\operatorname{Ext}(\hat{A}, R) = 0$, there is an extension $\bar{\theta} : F \to R$ of θ. Then let $f : B \to 2$,

$$f(w) = \Phi_{p,\mu}(\bar{\theta}(a) + pR) \text{ for } w = \langle \mu, p, a \rangle \text{ with } \mu \in R_0^r,\ p \in \mathbb{Z},\ a \in F.$$

Moreover, define $f^* : S \to \omega$ by

$$f^*(\gamma) = \min\{n \in \mathbb{N} | \forall m \geq n : (2[p_{\gamma,n}^{j_{p_{\gamma,n}}} \bar{\theta}(z_{\gamma,0})] + 1)^{2r+2} < p_{\gamma,m} \text{ and}$$
$$(2[p_{\gamma,n}^{j_{p_{\gamma,n}}} \bar{\theta}(y_{\gamma,j})] + 1)^{2r+2} < p_{\gamma,m} \text{ for } j = 1,\dots,r\}.$$

Then for all $n \geq f^*(\gamma)$ holds $f(\eta_\gamma(n)) = c_\gamma(n)$:
It is

$$\begin{aligned}\bar\theta(a'_{\gamma,n}) + p_{\gamma,n}R &= \theta(w_{\gamma,n}) - \bar\theta(z_{\gamma,0}) + \sum_{j=1}^{r}\mu_{\gamma,j}(n)\bar\theta(y_{\gamma,j}) + pR \\ &= k^{c_\gamma(n)}_{p_{\gamma,n}\mu_\gamma(n)} - \bar\theta(z_{\gamma,0}) + \sum_{j=1}^{r}\mu_{\gamma,j}(n)\bar\theta(y_{\gamma,j}) + pR\end{aligned}$$

and hence

$$\begin{aligned}f(\eta_\gamma(n)) &= \Phi_{p_{\gamma,n},\mu_\gamma(n)}(\bar\theta(a'_{\gamma,n}) + pR) \\ &= \Phi_{p_{\gamma,n},\mu_\gamma(n)}(k^{c_\gamma(n)}_{p_{\gamma,n}\mu_\gamma(n)} - \bar\theta(z_{\gamma,0}) + \sum_{j=1}^{r}\mu_{\gamma,j}(n)\bar\theta(y_{\gamma,j}) + pR) \\ &= c_\gamma(n).\end{aligned}$$

Then by Lemma 4.4.12 there is a ladder system which satisfies the 2–uniformization. This completes the proof in the first case.

In the second case we can assume without loss of generality that $R = R(p)$. Define a strictly increasing sequence of integers in the following way. Let $t_0 = 0$. If t_{i-1} is already defined, let $t_i = t_{i-1} + d_i$ where d_i is the smallest integer such that $(2p^{t_{i-1}} + 1)^{2r+2}p^{2t_{i-1}} < p_i^{d_i}$. Now, let $B = \bigoplus_\omega R_0^r \times \bigoplus_\omega F$ and $B_\nu = \bigoplus_\omega R_0^r \times \bigoplus_\omega(\bigoplus_{\beta<\nu} F_\beta)$. Define $\mu_\gamma(n) = \langle \mu_{\gamma,j}(m) : j = 1, \ldots, r, m < t_{n+1}\rangle >$ and $\eta_\gamma(n) = \left\langle \mu_\gamma(n), a'_{\gamma,m} : m < t_{n+1}\right\rangle$. By Lemma 4.4.11 there are integers $k^0_{n,\mu_\gamma(n)}$ and $k^1_{n,\mu_\gamma(n)}$ and a function $\Phi_{i,\mu_\gamma(n)} : R(p)/pR(p) \to 2$ with the property stated there. Let a 2–coloring $c = \{c_\gamma : \gamma \in S\}$ be given. Let $b_n = \max\{i \in \mathbb{N}_0 : t_i \leq n\}$ and define $\theta : K \to R$ by $\theta(w_{\gamma,n}) = k^{c_\gamma(b_n)}_{n,\mu_\gamma(n)}$. Then there is an extension $\bar\theta : F \to R$ of θ. For $w = \langle\langle \mu_j(m) : j = 1, \ldots, r\rangle, a'_m : m < t_i\rangle \in B$ let $f : B \to 2$,

$$f(w) = \Phi_{i,\mu}(\sum_{m<t_i} p^m\bar\theta(a'_m) + p^{t_i}R)$$

and $f^* : S \to \omega$,

$$f^*(\gamma) = \min\{n \in \mathbb{N} | \forall m \geq n : [\bar\theta(z_{\gamma,0})]_p^{m+1} \leq p^{t_m} \text{ and } [\bar\theta(y_{\gamma,j})]_p^{m+1} \leq p^{t_m} \text{ for all } j \leq r\}.$$

Then for all $n \geq f^*(\gamma)$ is $f(\eta_\gamma(n)) = c_\gamma(n)$:
We have

$$\bar\theta(a'_{\gamma,m}) = \theta(w_{\gamma,m}) - \bar\theta(z_{\gamma,m}) + p\bar\theta(z_{\gamma,m+1}) + \sum_{j=1}^{r}\mu_{\gamma,j}(m)\bar\theta(y_{\gamma,j})$$

and therefore,

$$\begin{aligned}\sum_{m<t_{n+1}} p^m \bar{\theta}(a'_{\gamma,m}) + p^{t_{n+1}}R = &\sum_{m<t_{n+1}} p^m k^{c_\gamma(b_m)}_{m,\mu_\gamma(m)} + \sum_{m<t_{n+1}} p^{m+1}\bar{\theta}(z_{\gamma,m+1}) \\ &- \sum_{m<t_{n+1}} p^m \bar{\theta}(z_{\gamma,m}) + \sum_{m<t_{n+1}}\sum_{j=1}^{r}\mu_{\gamma,j}(m)\bar{\theta}(y_{\gamma,j}) + p^{t_{n+1}}R \\ = &\sum_{m<t_n} p^m k^{c_\gamma(b_m)}_{m,\mu_\gamma(m)} + \sum_{m=t_n}^{t_{n+1}-1} p^m k^{c_\gamma(b_n)}_{m,\mu_\gamma(m)} - \bar{\theta}(z_{\gamma,0}) \\ &+ \sum_{m<t_{n+1}}\sum_{j=1}^{r}\mu_{\gamma,j}(m)\bar{\theta}(y_{\gamma,j}) + p^{t_{n+1}}R.\end{aligned}$$

Hence $f(\eta_\gamma(n)) = c_\gamma(n)$ and by Lemma 4.4.12 there is a ladder system which satisfies the 2–uniformization. □

Remark 4.4.14. *Theorem 4.4.13 also shows that if there exists a torsion–free non–free R_0–module of cardinality $\aleph_1$ which is a Whitehead module, then there is a ladder system which satisfies 2–uniformization.*

There is an immediate connection between the existence of Whitehead groups and the existence of R–Whitehead groups.

Theorem 4.4.15. *In every model of set theory the following holds.*
There exists a non–free Whitehead group of cardinality $\aleph_1$ if and only if there exists a non–R_0–free R–Whitehead group of cardinality $\aleph_1$.

Nevertheless, the following question remains unanswered.

Question. Is there a model of set theory in which there exists an R–Whitehead group that is not an R_0–Whitehead group?

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